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Construction of optimal controls for a distributed parameter control system

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Teoria dei controlli. — *Construction of optimal controls for a distributed parameter control system* (*). Nota di MEHMET NAMIK OĞUZTÖRELI (**), presentata (***) dal Socio M. PICONE.

RIASSUNTO. — Questo lavoro è dedicato alla dimostrazione dell'esistenza e dell'unicità di un controllo ottimo per un sistema lineare unidimensionale ben posto con parametro di controllo distribuito, mediante un integrale input–output in connessione con un funzionale quadratico. Ammissibili controlli soddisfano talune condizioni al contorno e equazioni integrali alle derivate parziali del primo ordine.

Per mezzo di un metodo di approssimazione trigonometrica si costruisce una soluzione ottimale. Controlli ottimali soddisfano una equazione integro differenziale di tipo ellittico. Unicità e controllo ottimale sono discussi.

I. DESCRIPTION OF THE CONTROL SYSTEM AND THE OPTIMIZATION PROBLEM.

Consider a well-posed one-dimensional linear control system defined on the spatial interval $0 \leq x \leq \pi$ in the processing time interval $0 \leq t \leq \pi$. Put

$$R = \{(t, x) \mid 0 \leq t \leq \pi, 0 \leq x \leq \pi\}. \tag{I.1}$$

The boundary of the square will be denoted by ∂R. We assume that the control system is described by an *input-output* relationship of the form

$$u(t, x) = \varphi(t, x) + \iint_R K(t, x; \tau, \xi)\, v(\tau, \xi)\, d\xi\, d\tau \tag{I.2}$$

for $(t, x) \in R$, where $v(t, x)$ is the input (*control*) variable at (t, x), $u(t, x)$ is the output variable at (t, x), $\varphi(t, x)$ is a given function absolutely continuous on R, and $K(t, x; \tau, \xi)$ is a given function on $R \times R$ which is absolutely continuous in (t, x) for almost all fixed (τ, ξ) and square integrable in (τ, ξ) for fixed (t, x).

A control $v = v(t, x)$ is assumed to be admissible if it is square integrable on R, and

$$v(0, x) = v(\pi, x) = 0 \text{ for } 0 \leq x \leq \pi, \quad \frac{\partial v(t, 0)}{\partial t} = \frac{\partial v(t, \pi)}{\partial t} = 0 \text{ for } 0 \leq t \leq \pi, \tag{I.3}$$

and

$$D(v) = \iint_R \left[\left(\frac{\partial v(t, x)}{\partial t}\right)^2 + \left(\frac{\partial v(t, x)}{\partial x}\right)^2\right] dx\, dt = 1. \tag{I.4}$$

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Note that $v(t, x) = 0$ is not an admissible control by virtue of the condition (I.4). We denote by V the set of all admissible controls.

The performance of the system S under an admissible control $v(t, x)$ is measured by the following cost functional

$$J(v) = \iint_R [u^2(t, x) + v^2(t, x)] \, dx \, dt. \tag{I.5}$$

In the present paper we are concerned with the finding of an admissible control $v^0 = v^0(t, x)$ for which the cost functional $J(v)$ assumes its minimum. Such a control will be called *optimal*.

It is quite clear that the specific form of R is not a restriction for the generality of the problem.

The above formulated optimization problem involves the minimization of the functional $J(v)$ on the space V subject to the constraint (I.4). Hence, for any optimal control, the functional

$$F(v) = \iint_R \left\{ u^2(t, x) + v^2(t, x) - \mu \left[\left(\frac{\partial v(t, x)}{\partial t} \right)^2 + \left(\frac{\partial v(t, x)}{\partial x} \right)^2 \right] \right\} dx \, dt \tag{I.5*}$$

assumes its minimum, where μ is the Lagrange multiplier, on the set of square integrable function $v(t, x)$ satisfying the boundary conditions (I.3), where $u = u(t, x)$ is given by (I.2).

The Euler–Lagrange equation associated with the functional $F(v)$ is the following integro-differential equation of elliptic type:

$$\mu\left(\frac{\partial^2 v(t, x)}{\partial t^2} + \frac{\partial^2 v(t, x)}{\partial x^2}\right) - v(t, x) = \varphi_1(t, x) + \iint_R K_1(t, x; \sigma, \eta) \, v(\sigma, \eta) \, d\eta \, d\sigma, \tag{I.6}$$

where

$$\varphi_1(t, x) = \iint_R K(\tau, \xi; t, x) \, \varphi(\tau, \xi) \, d\xi \, d\tau \tag{I.7}$$

and

$$K_1(t, x; \sigma, \eta) = \iint_R K(t, x; \tau, \xi) \, K(\tau, \xi; \sigma, \eta) \, d\xi \, d\tau. \tag{I.8}$$

Thus any optimal control satisfies the boundary conditions (I.3) and the integro-differential equation (I.6) with a specified value of μ.

II. CONSTRUCTION OF AN OPTIMAL CONTROL.

Consider the Hilbert space $V \equiv L^2(R)$ of functions $v = v(t, x)$ which are square integrable on R. Clearly $V \subset V$ and the functional $J(v)$ is strongly continuous on V. Obviously $J(v) > 0$ for all $v \in V$.

The sequence $\{\sin jt \sin kx\}$ $(j, k = 1, \cdots, n)$ is complete in V in the sense that for any $v \in V$ and any $\varepsilon > 0$ there exist a linear combination

$$v_n \equiv v_n(t, x) = \sum_{j,k=1}^{n} \alpha_{jk} \sin jt \sin kx, \tag{II.1}$$

such that $\| v_n - v \| < \varepsilon$, where n depends on ε, the α_{jk}'s are certain real numbers, and $\| \cdot \|$ denotes the norm in V.

Any linear combination $v_n(t, x)$ satisfies the boundary conditions (I.3). Hence, $v_n(t, x)$ is admissible, if it satisfies the condition (I.4). Thus, $v_n(t, x)$ is admissible, if

$$\text{(II.2)} \qquad \mathrm{D}(v_n) \equiv \frac{\pi^2}{4} \sum_{j,k=1}^{n} (j^2 + k^2)\, \alpha_{jk}^2 = 1 .$$

Put $\alpha = (\alpha_{jk})$, $j, k = 1, \cdots, n$, and $f_n(\alpha) = \mathrm{J}(v_n)$. Clearly the function $f_n(\alpha)$ is a positive definite quadratic form in the variables α_{jk}.

In terms of α_{jk}'s the optimization problem formulated in § I reduces to the minimization of the function $f_n(\alpha)$ on the surface S_{n^2} of the n^2–dimensional ellipsoid defined by Eq. (II. 2). Since the function $f_n(\alpha)$ is continuous on S_{n^2} and S_{n^2} is compact, $f_n(\alpha)$ achieves its minimum at some point $\alpha = \alpha^0$ of S_{n^2}. Consider the function

$$\text{(II.3)} \qquad v_n^0(t, x) = \sum_{j,k=1}^{n} \alpha_{jk}^0 \sin jt \sin kx ;$$

and put

$$\text{(II.4)} \qquad \mu_n^0 = f_n(\alpha^0) = \min_{\alpha \in \mathrm{S}_{n^2}} f_n(\alpha) = \mathrm{J}(v_n^0).$$

Clearly $\mu_n^0 > 0$ for all n. It can be easily seen that the sequence $\{\mu_n^0\}$ is monotonic decreasing. Therefore the limit

$$\text{(II.5)} \qquad \mu^0 = \lim_{n \to \infty} \mu_n^0 = \lim_{n \to \infty} \mathrm{J}(v_n^0)$$

exists and $\mu^0 > 0$.

Consider the sequence $\{v_n^0(t, x)\}$. Since the sequence $\{\mu_n^0\}$ is convergent, there exists a constant M_1 such that $0 < \mu_n^0 \le \mathrm{M}_1$ for all n. Thus

$$\text{(II.6)} \qquad \mathrm{J}(v_n^0) = \iint_{\mathrm{R}} \{[u_n^0(t, x)]^2 + [v_n^0(t, x)]^2\}\, \mathrm{d}x\, \mathrm{d}t \le \mathrm{M}_1 ,$$

where $u_n^0(t, x)$ is defined by (I.2) with $v = v^0(t, x)$. Hence

$$\text{(II.7)} \qquad \iint_{\mathrm{R}} [v_n^0(t, x)]^2\, \mathrm{d}x\, \mathrm{d}t \le \mathrm{M}_1$$

for all n, which shows that the sequence $\{v_n^0(t, x)\}$ is bounded in the Hilbert space $V = \mathrm{L}^2(\mathrm{R})$. Thus the sequence $\{v_n^0(t, x)\}$ is weakly compact in V. Accordingly, we can choose a subsequence from the sequence $\{v_n^0(t, x)\}$ which converges weakly to a $v^0(t, x) \in V$. We change the indices so that $\{v_n^0(t, x)\}$ will denote the weakly convergent subsequence:

$$\text{(II.8)} \qquad v_n^0(t, x) \xrightarrow{w} v^0(t, x).$$

We can easily show that the sequence

$$\{u_n^0(t,x) = \varphi(t,x) + \iint_R K(t,x;\tau,\xi)\, v_n^0(\tau,\xi)\, d\xi\, d\tau\} \tag{II.9}$$

is uniformly bounded and equicontinuous on R, and therefore, by Ascoli–Arzela's theorem, is compact. Thus we can select a uniformly convergent subsequence from $\{u_n^0(t,x)\}$ converging to a continuous function $u^0(t,x)$ on R. Changing the indices once more, so that $n = 1, 2, 3, \cdots$ refers to this uniformly convergent subsequence, we can write

$$u_n^0(t,x) \to u^0(t,x) = \varphi(t,x) + \iint_R K(t,x;\tau,\xi)\, v^0(\tau,\xi)\, d\xi\, d\tau \tag{II.10}$$

uniformly on R. By the help of the dominated convergence theorem, we obtain

$$J(v^0) = \lim_{n\to\infty} J(v_n^0) = \lim_{n\to\infty} \mu_n^0 = \mu^0, \tag{II.11}$$

which shows that the functional $J(v)$ achieves its minimum for $v = v^0(t,x)$, and μ^0 is the minimal value of $J(v)$.

We now show that $v^0(t,x)$ is the required optimal control. To do so we have to prove that $v^0(t,x)$ is admissible.

First of all, $v^0(t,x)$ clearly satisfies the boundary conditions $v^0(0,x) = v^0(\pi,x) = 0$ for $0 \le x \le \pi$. Further, let us note that, since $D(v_n^0) = 1$, we have

$$\iint_R \left[\frac{\partial v_n^0(t,x)}{\partial t}\right]^2 dx\, dt \le 1 , \quad \iint_R \left[\frac{\partial v_n^0(t,x)}{\partial x}\right]^2 dx\, dt \le 1, \tag{II.12}$$

so that the sequences $\left\{\frac{\partial v_n^0(t,x)}{\partial t}\right\}$ and $\left\{\frac{\partial v_n^0(t,x)}{\partial x}\right\}$ are bounded in $V = L^2(R)$,

and therefore they are weakly compact. Hence we can extract a weakly convergent subsequence, which we shall write by the same indices, from the above sequences approaching to certain functions $g(t,x)$ and $h(t,x)$ of V:

$$\frac{\partial v_n^0(t,x)}{\partial t} \xrightarrow{w} g(t,x) , \quad \frac{\partial v_n^0(t,x)}{\partial x} \xrightarrow{w} h(t,x). \tag{II.13}$$

Obviously $g(t,0) = g(t,\pi) = 0$ since $v_n^0(t,x)$'s are admissible. We now show that

$$g(t,x) = \frac{\partial v^0(t,x)}{\partial t} , \quad h(t,x) = \frac{\partial v^0(t,x)}{\partial x} . \tag{II.14}$$

To do so we proceed as follows:

As it is shown above the positive definite quadratic form $f_n(\alpha)$ achieves its minimum μ_n^0 subject to the constraint (I.4) at the point $\alpha^0 \in S_{n^2}$. By the method of Lagrange multipliers, we have

$$\text{(II.15)} \qquad \frac{\partial}{\partial \alpha_{jk}} \left\{ f_n(\alpha) - \mu_n^0 \iint_R \left[\left(\sum_{j,k=1}^{n} j\alpha_{jk} \cos jt \sin kx \right)^2 + \left(\sum_{j,k=1}^{n} k\alpha_{jk} \sin jt \cos kx \right)^2 \right] \right\} \mathrm{d}x\, \mathrm{d}t = 0$$

for $\alpha = \alpha^0$, which yields to the following n^2–equations:

$$\text{(II.16)} \qquad \iint_R \left\{ u_n^0(t, x) \frac{\partial u_n^0(t, x)}{\partial \alpha_{jk}} + v_n^0(t, x) \sin jt \sin kx - \mu_n^0 \left[\frac{\partial v_n^0(t, x)}{\partial t} j \cos jt \sin kx + \frac{\partial v_n^0(t, x)}{\partial x} k \sin jt \cos kx \right] \right\} \mathrm{d}x\, \mathrm{d}t = 0$$

for $j, k = 1, \cdots, n$, where

$$\text{(II.17)} \qquad \frac{\partial u_n^0(t, x)}{\partial \alpha_{jk}} = \iint_R \mathrm{K}(t, x; \tau, \xi) \sin j\tau \sin k\xi\, \mathrm{d}\xi\, \mathrm{d}\tau.$$

Multiplying each of the equations (II.16) by an arbitrary constant $\gamma_{jk}^{(n)}$ and summing over j and k from 1 to n, and manipulating on the first term, we obtain

$$\text{(II.18)} \qquad \iint_R \left\{ \left[\varphi_1(t, x) + \iint_R \mathrm{K}_1(t, x; \sigma, \eta)\, v_n^0(\sigma, \eta)\, \mathrm{d}\eta\, \mathrm{d}\sigma + v_n^0(t, x) \right] \mathrm{Q}_n(t, x) - \mu_n^0 \left[\frac{\partial v_n^0(t, x)}{\partial t} \frac{\partial \mathrm{Q}_n(t, x)}{\partial t} + \frac{\partial v_n^0(t, x)}{\partial x} \frac{\partial \mathrm{Q}_n(t, x)}{\partial x} \right] \right\} \mathrm{d}x\, \mathrm{d}t = 0,$$

where

$$\text{(II.19)} \qquad \mathrm{Q}_n(t, x) = \sum_{j,k=1}^{u} \gamma_{jk}^{(n)} \sin jt \sin kx.$$

Integrating by parts, we find

$$\text{(II.20)} \qquad \iint_R \left\{ \left[\varphi_1(t, x) + \iint_R \mathrm{K}_1(t, x; \sigma, \eta)\, v_n^0(\sigma, \eta)\, \mathrm{d}\eta\, \mathrm{d}\sigma + v_n^0(t, x) \right] \mathrm{Q}_n(t, x) - \mu_n^0\, v_n^0(t, x) \left[\frac{\partial^2 \mathrm{Q}_n(t, x)}{\partial t^2} + \frac{\partial^2 \mathrm{Q}_n(t, x)}{\partial x^2} \right] \right\} \mathrm{d}x\, \mathrm{d}t = 0.$$

Now, let $\mathrm{Q}(t, x)$ be an arbitrary function, twice continuously differentiable in R satisfying the boundary condition

$$\text{(II.21)} \qquad \mathrm{Q}(t, x) \big|_{\partial \mathrm{R}} = 0.$$

Clearly, we can choose the coefficients $\gamma_{jk}^{(n)}$ in such a way that

$$\lim_{n\to\infty} \iint_R | Q_n^{(i)}(t, x) - Q^{(i)}(t, x)|^2 \, dx \, dt = 0 \qquad (i = 0, 1, 2), \tag{II.22}$$

where $Q_n^{(i)}(t, x)$ and $Q^{(i)}(t, x)$ denote the partial derivatives of $Q_n(t, x)$ and $Q(t, x)$ with respect to t and x of order i, respectively. Since $\mu_n^0 \to \mu^0$, $v_n^0(t, x) \to v^0(t, x)$ weakly and $u_n^0(t, x) \to u^0(t, x)$ uniformly on R, we have

$$\iint_R \left\{ w(t, x) Q(t, x) - \mu^0 v^0(t, x) \left[\frac{\partial^2 Q(t, x)}{\partial t^2} + \frac{\partial^2 Q(t, x)}{\partial x^2} \right] \right\} dx \, dt = 0 \tag{II.23}$$

where

$$w(t, x) = \varphi_1(t, x) + \iint_R K_1(t, x; \sigma, \eta) v^0(\sigma, \eta) \, d\eta \, d\sigma + v^0(t, x). \tag{II.24}$$

Clearly, the function $w(t, x)$ is square integrable on R. Put

$$W(t, x) = \frac{1}{2\pi} \iint_R G(t, x; \tau, \xi) w(\tau, \xi) \, d\xi \, d\tau, \tag{II.25}$$

where $G(t, x; \tau, \xi)$ is the Green's function for the domain R. Hence we have

$$W(t, x)|_{\partial R} = 0 \qquad \text{and} \qquad \nabla^2 W(t, x) = w(t, x), \tag{II.26}$$

where ∇^2 is the Laplace operator in $(t, x) : \nabla^2 = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}$. Applying Green's formula to the first term of (II.23), and making use of the Eqs. (II.21), (II.25) and (II.26), we obtain

$$\iint_R \{ W(t, x) - \mu^0 v^0(t, x) \} \nabla^2 Q(t, x) \, dx \, dt = 0. \tag{II.27}$$

Clearly the function

$$\psi(t, x) = W(t, x) - \mu^0 v^0(t, x) \tag{II.28}$$

is square integrable on R. We now choose the function $Q(t, x)$ as follows

$$Q(t, x) = \frac{1}{2\pi} \iint_R \psi(\tau, \xi) G(t, x; \tau, \xi) \, d\xi \, d\tau, \tag{II.29}$$

which satisfies the boundary condition (II.21). Further we have $\nabla^2 Q(t, x) = \psi(t, x)$, which, on account of Eqs. (II.27), (II.28), yields

$$\iint_R \{ W(t, x) - \mu^0 v^0(t, x) \}^2 \, dx \, dt = 0. \tag{II.30}$$

Thus, $W(t,x) - \mu^0 v^0(t,x)$ vanishes almost everywhere in R.

$$\mu^0 v^0(t,x) = W(t,x) \quad \text{a.e.} \tag{II.31}$$

Since $W(t,x)$ is twice differentiable, almost everywhere in R, the function $v^0(t,x)$ is also twice differentiable almost everywhere in R. Further, making use of Eqs. (II.24) and (II.25), we find

$$\mu^0 \nabla^2 v^0(t,x) - v^0(t,x) = \\ = \varphi_1(t,x) + \iint_R K_1(t,x;\sigma,\eta)\, v^0(\sigma,\eta)\, d\eta\, d\sigma, \tag{II.32}$$

which is the Euler-Lagrange equation associated with the functional $F(v)$.

Note that, by virtue of Eq. (II.32), $\nabla^2 v^0(t,x)$ is square integrable on R, which implies the absolute continuity of the partial derivatives $\frac{\partial v^0(t,x)}{\partial t}$ and $\frac{\partial v^0(t,x)}{\partial x}$ on R. Since the function $v^0(t,x)$ is continuously differentiable on R, again by Eq. (II.32), $\nabla^2 v^0(t,x)$ is absolutely continuous on R.

We now establish the equalities (II.14). For this purpose consider Eq. (II.18) and pass to the limit as $n \to \infty$. Then, by virtue of Eqs. (II.5), (II.8) and (II.13), we obtain

$$\iint_R \left\{ w(t,x)\, Q(t,x) - \mu^0 \left[g(t,x) \frac{\partial Q(t,x)}{\partial t} + \right.\right. \\ \left.\left. + h(t,x) \frac{\partial Q(t,x)}{\partial x} \right] \right\} dx\, dt = 0, \tag{II.33}$$

where $Q(t,x)$ is an arbitrary function satisfying the boundary condition (II.21) and twice continuously differentiable in R, and $w(t,x)$ is given by Eq. (II.24). Further, integrating by parts in (II.23) and observing that $v^0(t,x)$ satisfies the boundary conditions (I.3), we find

$$\iint_R \left\{ w(t,x)\, Q(t,x) - \mu^0 \left[\frac{\partial v^0(t,x)}{\partial t} \frac{\partial Q(t,x)}{\partial t} + \right.\right. \\ \left.\left. + \frac{\partial v^0(t,x)}{\partial x} \frac{\partial Q(t,x)}{\partial x} \right] \right\} dx\, dt = 0, \tag{II.34}$$

or, subtracting (II.33) from (II.34),

$$\iint_R \left\{ \left[\frac{\partial v^0(t,x)}{\partial t} - g(t,x) \right] \frac{\partial Q(t,x)}{\partial t} + \right. \\ \left. + \left[\frac{\partial v^0(t,x)}{\partial x} - h(t,x) \right] \frac{\partial Q(t,x)}{\partial x} \right\} dx\, dt = 0. \tag{II.35}$$

Now, taking into account the arbitrariness of the function $Q(t,x)$, we easily establish Eqs. (II.14). Hence $v^0(t,x)$ is admissible. Since it also minimizes the functional $J(v)$ on the set V, it is optimal as asserted.

III. UNIQUENESS OF THE OPTIMAL CONTROLS.

So far we have only established the existence of an optimal control. We have also shown that an optimal control satisfies the integro-differential equation (II.32), where $\mu_0\,(>0)$ is the minimal value of the functional $J(v)$ in the space V. The uniqueness of the optimal controls is closely connected with the uniqueness of the solutions of Eq. (II.32) subject to the conditions (I.3) and (I.4). We now investigate briefly this problem.

In the previous section we constructed the solution $v^0(t,x)$ of Eq. (II.32) which satisfies the conditions (I.3) and (I.4). Suppose that Eq. (II.32) has another solution, say $v^1(t,x)$ which satisfies the same conditions. Consider the function

$$v(t,x)=v^0(t,x)-v^1(t,x). \tag{III.1}$$

Clearly we have

$$\begin{cases} v(0,x)=v(\pi,x)=0 & \text{for } 0\le x\le\pi,\\ v(t,0)=\psi_1(t)\ ,\quad v(t,\pi)=\psi_2(t) & \text{for } 0\le t\le\pi, \end{cases} \tag{III.2}$$

where

$$\begin{gathered}\psi_1(t)=v^0(t,0)-v^1(t,0)\ ,\quad \psi_2(t,x)=v^0(t,\pi)-v^1(t,\pi),\\ \psi_1(0)=\psi_1(\pi)=\psi_2(0)=\psi_2(\pi)=0.\end{gathered} \tag{III.3}$$

The function $v(t,x)$ satisfies the following homogeneous equation

$$\mu^0\,\nabla^2 v(t,x)-v(t,x)=\iint_R K_1(t,x;\sigma,\eta)\,v(\sigma,\eta)\,d\eta\,d\sigma \tag{III.4}$$

for $(t,x)\in R$. First we consider the case $\psi_1(t)=\psi_2(t)\equiv 0$. In this case the function $v(t,x)$ is the solution of the following homogeneous Fredholm integral equation:

$$\mu^0\,v(t,x)=\frac{1}{\pi}\iint_R G_1(t,x;\tau,\xi)\,v(\tau,\xi)\,d\xi\,d\tau \tag{III.5}$$

where

$$G_1(t,x;\tau,\xi)=G(t,x;\tau,\xi)+\iint_R G(t,x;\sigma,\eta)\,K(\sigma,\eta;\tau,\xi)\,d\eta\,d\sigma, \tag{III.6}$$

and $G(t,x;\tau,\xi)$ is the Green's function of the domain R. According to Fredholm's theorem, we have two possibilities: $\pi\mu^0$ is a regular value or an eigenvalue of the kernel $G_1(t,x;\tau,\xi)$. In the first case $v(t,x)\equiv 0$ is the only solution, and consequently, $v^1(t,x)\equiv v^0(t,x)$, i.e., the optimal control is unique. If $\pi\mu^0$ is an eigenvalue of the kernel $G_1(t,x;\tau,\xi)$, then there exists at least one non-trivial solution of the integral equation (III.5), in which case the uniqueness of the optimal controls is no longer valid.

We now consider the general case where $\psi_1(t) \not\equiv 0$ and/or $\psi_2(t) \not\equiv 0$:

Let $v^*(t, x)$ be the solution of the Laplace equation $\nabla^2 v(t, x) = 0$ satisfying the boundary conditions (III.2). Then, putting $v(t, x) = v^*(t, x) + v^{**}(t, x)$, we obtain the following non-homogeneous Fredholm integral equation for $v^{**}(t, x)$:

$$\mu^0 v^{**}(t, x) = g^*(t, x) + \frac{1}{\pi} \iint_R G_1(t, x; \tau, \xi)\, v^{**}(\tau, \xi)\, d\xi\, d\tau, \tag{III.7}$$

where $G_1(t, x; \tau, \xi)$ is defined by (III.6), and

$$g^*(t, x) = \iint_R G(t, x; \tau, \xi)\, v^*(\tau, \xi)\, d\xi\, d\tau. \tag{III.8}$$

If $g^*(t, x) \equiv 0$ and if $\pi\mu^0$ is a regular value of the kernel $G_1(t, x; \tau, \xi)$ then the uniqueness of the optimal control is assured by the Fredholm theory.

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