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Sturmian Theorems for Characteristic Initial Value Problems

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Matematica. — *Sturmian Theorems for Characteristic Initial Value Problems* (*). Nota (**) di KURT KREITH, presentata dal Socio M. PICONE.

RIASSUNTO. — Si stabilisce un teorema sturmiano di confronto fra due equazioni del tipo $u_{xy} + pu = 0$, considerando soluzioni delle equazioni che assumono valori prescritti su caratteristiche dell'equazione stessa. Si dà anche un teorema di oscillazione per soluzioni delle sopraddette equazioni soddisfacenti ad opportune condizioni di regolarità e infinitesime su una caratteristica.

In [1] the author considered functions $u(s, t)$ and $v(s, t)$ which are solutions of the hyperbolic differential equations

$$u_{tt} - u_{ss} + p(s, t)\, u = 0\,, \tag{1}$$

$$v_{tt} - v_{ss} + q(s, t)\, v = 0\,. \tag{2}$$

Considering s as a space variable and t as time, the functions $u(s, t)$ and $v(s, t)$ may be interpreted as representing the motion of two vibrating strings which are oscillating about the equilibrium lines $u = 0$ and $v = 0$ while subject to restoring forces pu and qv, respectively. If $q(s, t) \geq p(s, t)$, then the second string is subject to a greater restoring force and therefore should oscillate faster than the first. The precise description of such behavior is called a Sturmian theorem for hyperbolic equations. In [1] Sturmian theorems were established of initial boundary value problems corresponding to (1) and (2).

The purpose of this paper is to establish Sturmian theorems for (1) and (2) in the context of the theory of characteristic initial value problems. To that end it is convenient to effect a change of variables

$$x = s + t$$
$$y = -s + t$$

so that (1) and (2) become

$$u_{xy} + pu = 0\,, \tag{3}$$

$$v_{xy} + qv = 0\,, \tag{4}$$

respectively. It is assumed that $p(x, y)$ and $q(x, y)$ are continuous throughout the first quadrant of the (x, y)–plane and that u and v are C^2 solutions of (3) and (4) in the classical sense.

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One form of Sturmian theorem for characteristic initial value problems follows readily from comparison theorems for differential inequalities. In particular, we shall make use of the following result which is a special case of [2; 20 IV].

LEMMA. *Let* $u(x, y)$ *and* $v(x, y)$ *satisfy*

$$u_{xy} + pu = 0$$
$$v_{xy} + qv \leq 0$$

for $(x, y) \in R(\xi, \eta) \equiv \{(x, y) \mid 0 < x < \xi ; 0 < y < \eta\}$. *If*

$$v(\xi, 0) + v(0, \eta) - v(0, 0) < u(\xi, 0) + u(0, \eta) - u(0, 0)$$

then $v(x, y) < u(x, y)$ *in* $\overline{R(\xi, \eta)}$.

THEOREM 1. *Let* $u(x, y)$ *and* $v(x, y)$ *be solutions of* (3) *and* (4), *respectively, in a rectangle* $R(\xi, \eta)$ *for which*

(i) $u(x, y) > 0$ for $(x, y) \in R(\xi, \eta)$

(ii) $u(\xi, \eta) = 0$.

If $p(x, y) \leq q(x, y)$ *in* $R(\xi, \eta)$ *and*

$$v(\xi, 0) + v(0, \eta) - v(0, 0) < u(\xi, 0) + u(0, \eta) - u(0, 0),$$

then $v(x, y)$ *cannot remain nonnegative in* $R(\xi, \eta)$.

Proof. We write $q(x, y) = p(x, y) + \delta(x, y)$, where $\delta(x, y) \geq 0$ in $R(\xi, \eta)$. If $v(x, y) \geq 0$ in $R(\xi, \eta)$, then we have

$$v_{xy} + pv = -\delta v \leq 0,$$

so that by the Lemma $v(x, y) < u(x, y)$ in $\overline{R(\xi, \eta)}$. In particular, $v(\xi, \eta) \leq 0$, so that $v(x, y)$ becomes negative in $R(\xi, \eta)$.

Remarks.

1. An analogous result to Theorem 1 holds in case $u(x, y)$ is negative in $R(\xi, \eta)$.

2. Using a more general version of the Lemma, one can easily generalize Theorem 1 to nonlinear differential inequalities for u and v.

The difficulty with Theorem 1 is that it is local in character and does not provide a useful means of establishing oscillatory behavior in the entire first quadrant of the (x, y)–plane. The next theorem is not restricted in this way. Motivated by the fact that the characteristic initial value problem for the telegraph equation

$$\begin{aligned} &u_{xy} + \lambda u = 0 \\ &u(x, 0) = u(0, y) = 1 \end{aligned} \tag{5}$$

has as its solution $u(x, y) = J_0(2\sqrt{\lambda xy})$, we consider a 1–parameter family on nonintersecting curves C_r whose graphs in the first quadrant are given by $y = f_r(x)$. We assume that each $f_r(x)$ is continuously differentiable and strictly decreasing for $0 < x < \infty$ and that each C_r is asymptotic to the positive x and y axes.

Our comparison theorem will apply to solutions u and v of characteristic initial value problems insofar as we shall require some mild regularity conditions near the characteristics which are typical of stable solutions corresponding to initial data tending to zero along the characteristics. Specifically we shall say that v is u-regular if for any pair of nodal curves C_{r_1} and C_{r_2} of u,

$$\lim_{x\to\infty} \int_{f_{r_1}(x)}^{f_{r_2}(x)} |u(x,y)\, v_y(x,y)|\, dy = 0,$$

$$\lim_{y\to\infty} \int_{f_{r_1}^{-1}}^{f_{r_2}^{-1}(y)} |v(x,y)\, u_x(x,y)|\, dx = 0.$$

Making use of these regularity conditions it is possible to establish a Sturmian comparison theorem and an oscillation theorem for (4).

THEOREM 2. *Le u be a solution of* (3) *which vanishes on C_{r_1} and C_{r_2} and is of constant sign in the region* D *enclosed by C_{r_1} and C_{r_2}. If $q(x,y) \geq p(x,y)$, $q(x,y) \not\equiv p(x,y)$, then every u-regular solution v of* (4) *has a zero in* D.

Proof. Suppose v does not have a zero in D. Without loss of generality we may assume that u and v are both positive in D. Multiplying (3) by v and (4) by u and subtracting yields

$$vu_{xy} - uv_{xy} + (p - q)\, uv = 0$$

or

$$(vu_x)_y - (uv_y)_x = (q - p)\, uv \,. \tag{6}$$

Let $D_M = \{(x,y) \in D \mid 0 < x < M \text{ and } 0 < y < M\}$. Integrating (6) over D_M and applying Green's theorem yields

$$-\oint_{\partial D_M} vu_x\, dx + uv_y\, dy = \iint_{D_M} (q-p)\, uv\, dx\, dy > 0 \,.$$

In order to obtain the desired contradiction it is sufficient to show that

$$\lim_{M\to\infty} \oint_{\partial D_M} vu_x\, dx + uv_y\, dy \geq 0 \,. \tag{7}$$

To establish (7), suppose that $f_{r_1}(x) < f_{r_2}(x)$ and define

$$x_i = f_{r_i}(\mathrm{M}) ; \qquad i = 1, 2,$$

$$y_i = f_{r_i}^{-1}(\mathrm{M}) ; \qquad i = 1, 2,$$

Then the boundary integral in (7) can be written

$$\begin{aligned}(8)\qquad &\int_{x_1}^{\mathrm{M}} v(x, f_{r_1}(x))\, u_x(x, f_{r_1}(x))\, dx + \int_{y_1}^{y_2} u(\mathrm{M}, y)\, v_y(\mathrm{M}, y)\, dy \\ &- \int_{x_2}^{\mathrm{M}} v(x, f_{r_2}(x))\, u_x(x, f_{r_2}(x))\, dx - \int_{x_1}^{x_2} v(x, \mathrm{M})\, u_x(x, \mathrm{M})\, dx .\end{aligned}$$

The regularity conditions on u and v assure that the second and fourth integrals in (8) tend to zero as $\mathrm{M} \to \infty$. Furthermore since $u_x \geq 0$ on C_{r_1} and $u_x \leq 0$ on C_{r_2}, the first and third terms make non-negative contributions for all M. This shows that (7) is satisfied and completes the proof.

An oscillation theorem follows readily from Theorem 2. A solution of (4) is said to be *oscillatory at* ∞ if it has zeros in the first quadrant arbitrarily far from the origin.

THEOREM 3. *Suppose v is a solution of* (4), *where* $q(x, y) \geq \lambda (xy)^{\alpha}$ *for some constants* $\lambda > 0$ *and* $\alpha > -1$. *If v is u-regular with respect to* (10) *below, then v is oscillatory at* ∞.

Proof. Consider the characteristic initial value problem

$$(9)\qquad \begin{aligned} u_{xy} + \lambda (xy)^{\alpha} u &= 0, \\ u(x, 0) = u(0, y) &= 1, \end{aligned}$$

where $\lambda > 0$ and $\alpha > -1$. Assuming a solution of the form $u(x, y) = u(z)$ where $z = xy$, (9) becomes

$$zu'' + u' + \lambda z^{\alpha} u = 0$$

$$u(0) = 1$$

which has the solution

$$u = \mathrm{J}_0\left(\beta z^{\frac{\alpha+1}{2}}\right)$$

where $\beta = \dfrac{2\lambda^{1/2}}{\alpha+1}$. Thus (9) has the solution

$$(10)\qquad u(x, y) = \mathrm{J}_0\left(\frac{2\lambda^{1/2}}{\alpha+1} (xy)^{\frac{\alpha+1}{2}}\right)$$

whose nodal lines C_{r_n} are determined by

$$xy = \left(\frac{\alpha+1}{2\lambda^{1/2}} j_n\right)^{\frac{2}{\alpha+1}},$$

where j_n is the n^{th} zero of J_0. It follows from Theorem 2 that every u–regular solution of (4) is oscillatory at ∞.

In order to apply Theorems 2 and 3 it is necessary to establish the existence of a class of u–regular functions. The following theorem makes precise our earlier assertion that the regularity conditions are realized by stable solutions corresponding to initial data tending to zero along the characteristics.

THEOREM 4. *Suppose $q(x, y)$ is bounded and that $v(x, y)$ is a solution of (4). If there exists an $\varepsilon > 0$ such that*

$$\lim_{x \to \infty} |v(x, y)| = 0 \quad \text{uniformly for} \quad 0 \leq y \leq \varepsilon, \tag{11}$$

$$\lim_{y \to \infty} |v(x, y)| = 0 \quad \text{uniformly for} \quad 0 \leq x \leq \varepsilon, \tag{12}$$

then v is u–regular with respect to (10).

Proof. Let C_{r_1} and C_{r_2} be nodal curves for (10) so that the C_{r_i} are given by $y = k_i/x$ for $i = 1, 2$ and $k_2 > k_1 > 0$. Let K be a bound for $|q(x, y)|$. From (4) it follows that

$$v_y(x, y) = -\int_0^x q(\xi, y)\, v(\xi, y)\, d\xi$$

and

$$|v_y(x, y)| \leq K \int_0^x |v(\xi, y)|\, d\xi .$$

By (11), $\lim_{x \to \infty} |v_y(x, y)| = o(x)$ uniformly in y, and since the function $u(x, y)$ given by (10) is bounded,

$$\lim_{x \to \infty} \int_{k_1/x}^{k_2/x} |u(x, y)\, v_y(x, y)|\, dy = 0 .$$

To establish the second condition of u–regularity we note that

$$u_x(x, y) = -\frac{\lambda^{1/2}}{x} J_1\left(\frac{2\lambda^{1/2}}{\alpha + 1} (xy)^{\frac{\alpha+1}{2}}\right) \cdot (xy)^{\frac{\alpha+1}{2}} .$$

For $k_1 < xy < k_2$ we have

$$|u_x(x, y)| = O(y)$$

so that the assumption (12) implies the second regularity condition

$$\lim_{y \to \infty} \int_{k_1/y}^{k_2/y} |v(x, y)\, u_x(x, y)|\, dx = 0 .$$

Remarks.

1. While it is customary to discuss Sturmian theorems for linear equations, no essential use of linearity was made above. Thus, for example, it is possible to consider $p = p(x, y, u)$ and $q = q(x, y, v)$ so long as the inequalities hypothesized for q and p hold for all values of x, y, u and v.

2. It is also possible to replace equations (3) and (4) by inequalities of the form.

$$uu_{xy} + pu^2 \geq 0 , \tag{3'}$$

$$vv_{xy} + qv^2 \leq 0 , \tag{4'}$$

In this case equation (6) becomes

$$(vu_x)_y - (uv_y)_x \geq (q - p)\, uv , \tag{6'}$$

which also leads to the desired contradiction.

3. It is essential to impose some sort of regularity conditions on solutions of (4) in order for Theorem 2 to be valid. To illustrate this fact, consider the function e^{x-y} which is a solution of (4) with $q \equiv 1$ but does not have a zero in any nodal domain defined by the telegraph equation (5).

References.

[1] K. Kreith, *Sturmian Theorems for Hyperbolic Equations*, « Proc. Amer. Math. Soc. », *22*, 277–281 (1969).

[2] W. Walter, *Differential – und Integral – Ungleichungen*, Springer 1964.