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A Further Generalization of the Second Isomorphism Theorem in Group Theory

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *A Further Generalization of the Second Isomorphism Theorem in Group Theory.* Nota (*) di OLAF TAMASCHKE, presentata dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — Sia $\mathbf{T}$ un semigruppo di Schur sul gruppo G. I sottogruppi H e K di G soddisfacciano alla $HK = KH$; e K ed HK siano entrambi $\mathbf{T}$–sottogruppi di G (risultino cioè unioni di $\mathbf{T}$–classi di G). In queste condizioni $\mathbf{T}$ induce un semigruppo di Schur $(\mathbf{T}_{HK})_{HK/K}$ su HK, semigruppo che (con riferimento alla moltiplicazione fra complessi) è generato dagli insiemi del tipo $K\mathfrak{C}K$, con $\mathfrak{C}$ variabile nella totalità delle $\mathbf{T}$–classi di G contenute in HK. In questa Nota sarà dimostrato che gli insiemi del tipo $K\mathfrak{C}K \cap H$, con $\mathfrak{C}$ variabile in quella tal totalità, generano (con riferimento alla moltiplicazione fra complessi) un semigruppo di Schur, $\boldsymbol{\Sigma}$, su H; che $\varphi : Y \to Y \cap H$ $(Y \in (\mathbf{T}_{HK})_{HK/K})$ fornisce una trasformazione isomorfa avente per dominio $(\mathbf{T}_{HK})_{HK/K}$, semigruppo di Schur su HK, e per codominio $\boldsymbol{\Sigma}$, semigruppo di Schur su H; e che $\psi : X \to XK$ $(X \in \boldsymbol{\Sigma})$ è l'inversa di φ. Il significato di questo teorema consiste in ciò, che nella situazione descritta un semigruppo di Schur su un gruppo « grande » HK può essere sostituito con una sua immagine isomorfa, semigruppo di Schur sul gruppo H « più piccolo » e perciò spesso più semplice nella sua struttura. Se per $\mathbf{T}$ si sceglie il gruppo G e per K un sottogruppo normale di G, il risultato precedente si riduce al secondo teorema sugli isomorfismi nella teoria dei gruppi.

The intention of introducing the concept of a Schur-semigroup in the theory of groups (cf. [1], [2] and [8], Chapter III) was not only that it might lead to a new differentiation in the structure of groups but also that it might make applicable to group theory various methods and results from the algebraic theory of semigroups.

(*) Pervenuta all'Accademia il 26 luglio 1969.

We can expect that the ease of dealing with a Schur-semigroup is directly related to the intricacy of the structure of the underlying group. Therefore theorems will be useful which allow an isomorphism of a Schur-semigroup on one group onto a Schur-semigroup on another group whose structure can be supposed to be less complicated. Such a situation is given if the underlying group of a Schur-semigroup is factorized in a certain way, and its investigation is the object of this note. This generalizes earlier results of [5] where we dealt with the special, though most important, case of double coset Schur-semigroups. (For the meaning of the double coset Schur-semigroups in the theory of permutation groups we refer to [6], Section 12, and to [8], Chapter IV). To make this paper self-contained we briefly recall some basic definitions from the theory of Schur-semigroups (cf. [1], [2] or [8], Chapter III).

Let G be a group. The set $\bar{G} := \{X \mid \varnothing \neq X \subseteq G\}$ is a semigroup with respect to subset multiplication (frequently called " complex " multiplication)

$$(X, Y) \to XY := \{xy \mid x \in X \text{ and } y \in Y\}.$$

DEFINITION 1. *A subsemigroup* **T** *of* $\bar{G}$ *is called a* Schur-semigroup *on* G *if it has a unit element and if there exists a set* $\mathfrak{T} \subseteq \bar{G}$ *such that*:

(1) $$G = \bigcup_{\mathfrak{C} \in \mathfrak{T}} \mathfrak{C}.$$

(2) $$\mathfrak{S} = \mathfrak{C} \quad or \quad \mathfrak{S} \cap \mathfrak{C} = \varnothing \quad for\ all \quad \mathfrak{S}, \mathfrak{C} \in \mathfrak{T}.$$

(3) $$\mathfrak{C}^{-1} := \{g^{-1} \mid g \in \mathfrak{C}\} \in \mathfrak{T} \quad for\ all \quad \mathfrak{C} \in \mathfrak{T}.$$

(4) $$X = \bigcup_{\substack{\mathfrak{C} \in \mathfrak{T} \\ \mathfrak{C} \cap X \neq \varnothing}} \mathfrak{C} \qquad for\ all \quad X \in \mathbf{T}.$$

(5) **T** *is generated by* $\mathfrak{T}$, *that is every element of* **T** *is the product of a finite number of elements of* $\mathfrak{T}$.

Since $\mathfrak{T}$ is uniquely determined by **T** and the axioms (1) to (5) we call the elements of $\mathfrak{T}$ the **T**–*classes* of G. We denote by $\mathfrak{C}_g$ the unique **T**–class containing $g \in G$.

A subgroup H of G is called a **T**–*subgroup* of G if H is the set theoretical union of **T**–classes, that is

$$H = \bigcup_{h \in H} \mathfrak{C}_h .$$

Every **T**–subgroup H of G defines two Schur-semigroups, namely

1. the Schur-semigroup $\mathbf{T}_H$ on H which is generated by all $\mathfrak{C}_h$ with $h \in H$,
2. the Schur-semigroup $\mathbf{T}_{G/H}$ on G which is generated by all $H\mathfrak{C}H$ with $\mathfrak{C} \in \mathfrak{T}$.

A $\mathbf{T}$–subgroup K of G is called $\mathbf{T}$–*normal* if

$$K\mathfrak{T} = \mathfrak{T}K \quad \text{holds for all} \quad \mathfrak{T} \in \mathfrak{T}.$$

Let F be a group, $\mathbf{\Sigma}$ a Schur-semigroup on F, and $\mathfrak{S}$ the set of all $\mathbf{\Sigma}$–classes of F (that is $\mathfrak{S}$ plays the same role for $\mathbf{\Sigma}$ as $\mathfrak{T}$ does for $\mathbf{T}$).

DEFINITION 2. *A mapping* φ *of* $\mathbf{T}$ *into* $\mathbf{\Sigma}$ *is called a* homomorphism *of the Schur-semigroup* $\mathbf{T}$ *on* G *into the Schur-semigroup* $\mathbf{\Sigma}$ *on* F *if it has the following properties.*

(1) $$(XY)^{\varphi} = X^{\varphi} Y^{\varphi} \quad \textit{for all} \quad X, Y \in \mathbf{T}.$$

(2) $$X^{\varphi} = \bigcup_{x \in X} \mathfrak{T}_x^{\varphi} \quad \textit{for all} \quad X \in \mathbf{T}.$$

(3) *For every* $\mathbf{T}$–*class* $\mathfrak{T}$ *of* G *there exists a* $\mathbf{\Sigma}$–*class* $\mathfrak{S}$ *of* F *such that*

$$\mathfrak{T}^{\varphi} = \mathfrak{S} \quad \textit{and} \quad (\mathfrak{T}^{-1})^{\varphi} = \mathfrak{S}^{-1}.$$

A homomorphism $\varphi : \mathbf{T} \to \mathbf{\Sigma}$ is called an *isomorphism* if φ is a bijective mapping.

Now we start on our investigations with the main theorem of this paper.

THEOREM 1. *Let* $\mathbf{T}$ *be a Schur-semigroup on the group* G. *Let* H *and* K *be subgroups of* G *such that* HK = KH. *Assume further that* K *and* HK *are* $\mathbf{T}$–*subgroups of* G. *Then*

(1) *The semigroup* $\mathbf{\Sigma}$, *which is generated (with respect to complex multiplication) by the set of all* $\mathfrak{S}_h := K\mathfrak{T}_h K \cap H$ *with* $h \in H$, *is a Schur-semigroup on* H, *and the* $\mathfrak{S}_h$, $h \in H$, *are the* $\mathbf{\Sigma}$–*classes of* H.

(2) $KX = XK$ *for all* $X \in \mathbf{\Sigma}$.

(3) *The mapping*

$$\varphi : Y \to Y \cap H \qquad (Y \in (\mathbf{T}_{HK})_{HK/K})$$

is an isomorphism of the Schur-semigroup $(\mathbf{T}_{HK})_{HK/K}$ *on* HK, *which is generated (with respect to complex multiplication) by the set of all* $K\mathfrak{T}_g K$ *with* $g \in HK$, *onto the Schur-semigroup* $\mathbf{\Sigma}$ *on* H. *The mapping*

$$\psi : X \to XK \qquad (X \in \mathbf{\Sigma})$$

is the inverse mapping of φ, *and hence is an isomorphism of the Schur-semigroup* $\mathbf{\Sigma}$ *on* H *onto the Schur-semigroup* $(\mathbf{T}_{HK})_{HK/K}$ *on* HK.

Remarks. – 1. If in the above theorem the Schur-semigroup $(\mathbf{T}_{HK})_{HK/K}$ on HK is considered as a " factor structure " of HK modulo K then Theorem 1 states that there exists a " factor structure " of H modulo $H \cap K = \mathfrak{S}_1$ which is isomorphic to the first, namely the Schur-semigroup $\mathbf{\Sigma}$ on H. Thus the situation of a Second Isomorphism Theorem is given. In fact, the Second Isomorphism Theorem in group theory is a special case of Theorem 1 if we apply it to $\mathbf{T} = G$ and to a normal subgroup K of G.

2. An analogous theorem holds for Schur-rings on finite groups instead of Schur-semigroups. It will be stated and proved in [9].

Proof. I. Since the $K\mathcal{C}_g K$, $g \in G$, are the $\mathbf{T}_{G/K}$–classes of G ([2], Proposition 1.4 (2)) for the sets $\mathcal{S}_h = K\mathcal{C}_h K \cap H$, $h \in H$, the following hold.

1. $$H = \bigcup_{h \in H} \mathcal{S}_h .$$

2. $$\mathcal{S}_g = \mathcal{S}_h \quad \text{or} \quad \mathcal{S}_g \cap \mathcal{S}_h = \varnothing \qquad \text{for all } g, h \in H .$$

3. $$\mathcal{S}_h^{-1} = \mathcal{S}_{h^{-1}} \qquad \text{for all} \quad h \in H .$$

Hence for the semigroup $\boldsymbol{\Sigma}$ which is generated (with respect to complex multiplication) by all the $\mathcal{S}_h$, $h \in H$, the properties (1), (2), (3), (5) of Definition 1 are satisfied by $\mathfrak{S} := \{\mathcal{S}_h \mid h \in H\}$.

In order to prove (4) of Definition 1 for $\boldsymbol{\Sigma}$ and $\mathfrak{S}$ (instead of $\mathbf{T}$ and $\mathfrak{T}$) we observe that every $g \in HK$ can be written as $g = hk$ with $h \in H$ and $k \in K$. Because of $h \in K\mathcal{C}_g K$ and the properties of the $\mathbf{T}_{G/K}$–classes of G (cf. [2], Proposition 1.4 (2)) we have $K\mathcal{C}_h K = K\mathcal{C}_g K$. Furthermore, each product $(K\mathcal{C}_x K)(K\mathcal{C}_y K)$ is the union of $\mathbf{T}_{G/K}$–classes, and if we choose $x, y \in H$ these $\mathbf{T}_{G/K}$–classes have the form $K\mathcal{C}_z K$ with $z \in H$.

It was shown in [5], p. 136, that

(i) $$(X \cap H)(Y \cap H) = XY \cap H \quad \textit{for all} \quad \varnothing \neq X, Y \subseteq HK$$
$$\textit{such that} \quad KXK = X \quad \textit{and} \quad KYK = Y.$$

Setting $X = K\mathcal{C}_x K$ and $Y = K\mathcal{C}_y K$ with $x, y \in H$, we obtain

$$\mathcal{S}_x \mathcal{S}_y = (K\mathcal{C}_x K)(K\mathcal{C}_y K) \cap H = \bigcup_{z \in (K\mathcal{C}_x K)(K\mathcal{C}_y K)} K\mathcal{C}_z K \cap H = \bigcup_{z \in (K\mathcal{C}_x K)(K\mathcal{C}_y K)} \mathcal{S}_z .$$

It follows that every element of $\boldsymbol{\Sigma}$ is the set theoretical union of elements of $\mathfrak{S}$, and therefore (4) of Definition 1 holds for $\boldsymbol{\Sigma}$ and $\mathfrak{S}$ (instead of $\mathbf{T}$ and $\mathfrak{T}$). Hence $\boldsymbol{\Sigma}$ is a Schur-semigroup on H.

II. In [5], p. 136, it was also shown that

$$K(Y \cap H) = Y \quad \text{for all} \quad \varnothing \neq Y \subseteq HK \quad \text{such that} \quad KY = Y.$$

Similarly one proves

(ii) $$(Y \cap H) K = Y \quad \textit{for all} \quad \varnothing \neq Y \subseteq HK \quad \textit{such that} \quad YK = Y.$$

In particular

$$K(Y \cap H) = Y = (Y \cap H) K \quad \text{for all} \quad \varnothing \neq Y \subseteq HK \quad \text{such that} \quad KYK = Y.$$

Since every element $X \in \Sigma$ can be written as

$$X = \mathcal{S}_{h_1} \cdots \mathcal{S}_{h_r} = ((K\mathcal{C}_{h_1}K) \cdots (K\mathcal{C}_{h_r}K)) \cap H \qquad (h_1, \cdots, h_r \in H)$$

we obtain $KX = XK$ for all $X \in \Sigma$.

III. Since every element $Y \in (\mathbf{T}_{HK})_{HK/K}$ can be written as

$$Y = (K\mathcal{C}_{h_1}K) \cdots (K\mathcal{C}_{h_r}K) \qquad (h_1, \cdots, h_r \in H)$$

equation (i) shows that

$$Y \cap H = \mathcal{S}_{h_1} \cdots \mathcal{S}_{h_r}$$

is an element of Σ. Conversely, every element $X = \mathcal{S}_{h_1} \cdots \mathcal{S}_{h_r}$ of Σ can be written as $X = Y \cap H$ with $Y = (K\mathcal{C}_{h_1}K) \cdots (K\mathcal{C}_{h_r}K) \in (\mathbf{T}_{HK})_{HK/K}$. Therefore

$$\varphi : Y \to Y \cap H \qquad (Y \in (\mathbf{T}_{HK})_{HK/K})$$

is a surjective mapping of $(\mathbf{T}_{HK})_{HK/K}$ onto Σ which, using (i) once again, satisfies Definition 2 (with H instead of F). Hence φ is a homomorphism of the Schur-semigroup $(\mathbf{T}_{HK})_{HK/K}$ on HK onto the Schur-semigroup Σ on H. Since φ is a surjective mapping (ii) shows that

$$\psi : X \to XK \qquad (X \in \Sigma)$$

is a mapping of Σ into $(\mathbf{T}_{HK})_{HK/K}$ such that both $\varphi\psi$ and $\psi\varphi$ are identity mappings. Hence φ and ψ are bijective mappings. Therefore they are isomorphisms of the relevant Schur-semigroups, and Theorem 1 is proved.

Particularly interesting is the case where H is a subgroup and K is a $\mathbf{T}$–subgroup of G such that $G = HK$. Furthermore: the $\mathbf{T}$–class $\mathcal{C}_1$ which contains the unit element $1 \in G$ is a subgroup ([2], Lemma 1.2 (2)), and hence it is a $\mathbf{T}$–subgroup, and even a $\mathbf{T}$–normal subgroup of G. For any subgroup H of G such that $G = H\mathcal{C}_1$, Theorem 1 shows that the Schur-semigroup $\mathbf{T}$ on G is isomorphic to a Schur-semigroup Σ on H. The hypotheses of this statement are satisfied for any transitive permutation group G and any transitive subgroup H of G if we take for $\mathbf{T}$ the double coset Schur-semigroup G/G_α where G_α is the stabilizer in G of a letter α. If the transitive subgroup H is even a regular subgroup of G then we obtain the Schur-semigroup version of Schur's theorem of the " transitivity module " of G_α on H (cf. [5], pp. 140–141).

We return to Theorem 1. Apart from the permutability with the $\mathbf{T}$–subgroup K we have assumed nothing of the subgroup H of G except that HK is a $\mathbf{T}$–subgroup. Let us look at the special case where both

H *and* K *are* $\mathbf{T}$*–subgroups of* G *such that* $HK = KH$.

Then $H \cap K$ and HK are also $\mathbf{T}$–subgroups ([2], Theorem 1.5), and the semigroup $(\mathbf{T}_H)_{H/H \cap K}$ which is generated by all the sets $(H \cap K)\, \mathcal{C}_h\, (H \cap K)$

with $h \in \mathrm{H}$ is a Schur-semigroup on H ([2], Proposition 1.4). For the Schur-semigroup $\boldsymbol{\Sigma}$ on H, defined by Theorem 1, each $\boldsymbol{\Sigma}$–class $\mathcal{S}_h = \mathrm{K}\mathcal{T}_h\mathrm{K} \cap \mathrm{H}$ of H is, under the present assumption, the union of **T**–classes of G:

$$\mathcal{S}_h = \bigcup_{x \in \mathcal{S}_h} \mathcal{T}_x \qquad (h \in \mathrm{H}) .$$

Each $\mathcal{S}_h$ is invariant under all left and right multiplications by all the elements of $\mathrm{H} \cap \mathrm{K}$. Hence we also have

$$\mathcal{S}_h = \bigcup_{x \in \mathcal{S}_h} (\mathrm{H} \cap \mathrm{K})\, \mathcal{T}_x \,(\mathrm{H} \cap \mathrm{K}) \qquad (h \in \mathrm{H}) .$$

Therefore each $\boldsymbol{\Sigma}$–class, even if it is not an element of the Schur-semigroup $(\mathbf{T}_{\mathrm{H}})_{\mathrm{H}/\mathrm{H}\cap\mathrm{K}}$, is at least an element of the set theoretical closure

$$\overline{(\mathbf{T}_{\mathrm{H}})}_{\mathrm{H}/\mathrm{H}\cap\mathrm{K}} : = \{\varnothing \neq \mathrm{X} \subseteq \mathrm{H} \mid \mathrm{X} = \bigcup_{x \in \mathrm{X}} (\mathrm{H} \cap \mathrm{K})\, \mathcal{T}_x \,(\mathrm{H} \cap \mathrm{K})\} ,$$

which means

$$\boldsymbol{\Sigma} \subseteq \overline{(\mathbf{T}_{\mathrm{H}})}_{\mathrm{H}/\mathrm{H}\cap\mathrm{K}} .$$

Under which conditions do we have the equation

$$\boldsymbol{\Sigma} = (\mathbf{T}_{\mathrm{H}})_{\mathrm{H}/\mathrm{H}\cap\mathrm{K}} \ ?$$

THEOREM 2. *Let* **T** *be a Schur-semigroup on* G, *and let both* H *and* K *be* **T**–*subgroups of* G *such that* $\mathrm{HK} = \mathrm{KH}$. *We denote by* $\boldsymbol{\Sigma}$ *the Schur-semigroup on* H *defined by Theorem* 1. *Then the following statements are equivalent.*

$$(1) \qquad \boldsymbol{\Sigma} = (\mathbf{T}_{\mathrm{H}})_{\mathrm{H}/\mathrm{H}\cap\mathrm{K}} .$$

$$(2) \qquad \mathrm{KX} = \mathrm{XK} \quad \textit{for all} \quad \mathrm{X} \in (\mathbf{T}_{\mathrm{H}})_{\mathrm{H}/\mathrm{H}\cap\mathrm{K}} .$$

Proof. (1) implies (2) by Theorem 1 (2). Assume that (2) holds. Every element $\mathrm{X} \in (\mathbf{T}_{\mathrm{H}})_{\mathrm{H}/\mathrm{H}\cap\mathrm{K}}$ has the form

$$\mathrm{X} = (\mathrm{H} \cap \mathrm{K})\, \mathcal{T}_{h_1} (\mathrm{H} \cap \mathrm{K}) \cdots (\mathrm{H} \cap \mathrm{K})\, \mathcal{T}_{h_r} (\mathrm{H} \cap \mathrm{K})$$

with $h_1, \cdots, h_r \in \mathrm{H}$. From the permutability property (2) of K we obtain

$$\mathrm{XK} = (\mathrm{K}\mathcal{T}_{h_1}\mathrm{K}) \cdots (\mathrm{K}\mathcal{T}_{h_r}\mathrm{K}) \in (\mathbf{T}_{\mathrm{HK}})_{\mathrm{HK}/\mathrm{K}} .$$

Since every element $\mathrm{Y} \in (\mathbf{T}_{\mathrm{HK}})_{\mathrm{HK}/\mathrm{K}}$ has the form

$$\mathrm{Y} = (\mathrm{K}\mathcal{T}_{h_1}\mathrm{K}) \cdots (\mathrm{K}\mathcal{T}_{h_r}\mathrm{K}) = (\mathrm{H} \cap \mathrm{K})\, \mathcal{T}_{h_1} (\mathrm{H} \cap \mathrm{K}) \cdots (\mathrm{H} \cap \mathrm{K})\, \mathcal{T}_{h_r} (\mathrm{H} \cap \mathrm{K})\mathrm{K}$$

with $h_1, \cdots, h_r \in \mathrm{H}$ the correspondence

$$\chi : \mathrm{X} \to \mathrm{XK} \qquad (\mathrm{X} \in (\mathbf{T}_{\mathrm{H}})_{\mathrm{H}/\mathrm{H}\cap\mathrm{K}})$$

is a surjective mapping of $(\mathbf{T}_H)_{H/H\cap K}$ onto $(\mathbf{T}_{HK})_{HK/K}$. Taking into account that the

$$(H \cap K)\, \mathfrak{C}_h\, (H \cap K) \qquad (h \in H)$$

are the $(\mathbf{T}_H)_{H/H\cap K}$–classes of H and that the

$$K \mathfrak{C}_h K \qquad (h \in H)$$

are the $(\mathbf{T}_{HK})_{HK/K}$–classes of HK, it is easy to check that χ satisfies Definition 2, and hence is a Schur-semigroup homomorphism. The kernel Ker χ ([2], Definition 2.6) is the set theoretical union of all those $(\mathbf{T}_H)_{H/H\cap K}$–classes $(H \cap K)\, \mathfrak{C}_h\, (H \cap K)$ of H which χ maps onto the unit element K of $(\mathbf{T}_{HK})_{HK/K}$, that is $\operatorname{Ker}\chi = H \cap K$. Therefore $\operatorname{Ker}\chi$ is the $(\mathbf{T}_H)_{H/H\cap K}$–class of H which contains the unit element $1 \in H$. By [2], Proposition 2.11, χ is an injective mapping, and hence χ is an isomorphism of the Schur-semigroup $(\mathbf{T}_H)_{H/H\cap K}$ on H onto the Schur-semigroup $(\mathbf{T}_{HK})_{HK/K}$ on HK. On the other hand

$$\varphi : Y \rightarrow Y \cap H \qquad (Y \in (\mathbf{T}_{HK})_{HK/K})$$

is an isomorphism of the Schur-semigroup $(\mathbf{T}_{HK})_{HK/K}$ on HK onto the Schur-semigroup $\mathbf{\Sigma}$ on H by Theorem 1 (3). It follows that $\chi\varphi$ is an isomorphism of the Schur-semigroup $(\mathbf{T}_H)_{H/H\cap K}$ on H onto the Schur-semigroup $\mathbf{\Sigma}$ on H. This implies that

$$(H \cap K)\, \mathfrak{C}_h\, (H \cap K)^{\chi\varphi} = K \mathfrak{C}_h K \cap H = \mathfrak{S}_h \quad \text{for all} \quad h \in H\,.$$

But we have

$$\mathfrak{S}_h = \bigcup_{x \in \mathfrak{S}_h} (H \cap K)\, \mathfrak{C}_x\, (H \cap K)$$

and we also have

$$(H \cap K)\, \mathfrak{C}_x\, (H \cap K)^{\chi\varphi} = \mathfrak{S}_h \quad \text{for all} \quad x \in \mathfrak{S}_h\,.$$

Therefore, by the bijectivity of $\chi\varphi$,

$$(H \cap K)\, \mathfrak{C}_h\, (H \cap K) = \mathfrak{S}_h \quad \text{for all} \quad h \in H\,.$$

From this we obtain $\mathbf{\Sigma} = (\mathbf{T}_H)_{H/H\cap K}$, and Theorem 2 is proved.

The permutability condition (2) of Theorem 2 is satisfied if K is a $\mathbf{T}_{HK}$–normal subgroup of HK, and in particular if K is a $\mathbf{T}$–normal subgroup of G. In this last case Theorem 1 becomes the Second Isomorphism Theorem for Schur-semigroups ([2], Theorem 2.13).

References.

[1] Tamaschke O., *An extension of group theory*. *Istituto Nazionale di Alta Matematica.* « Symposia Mathematica », *1*, 5–13 (1968).

[2] Tamaschke O., *An extension of group theory to S–semigroups*, « Math. Zeitschrift », *104*, 74–90 (1968).

[3] Tamaschke O., *A generalization of subnormal subgroups*, « Archiv d. Math. », *19*, 337–347 (1968).

[4] Tamaschke O., *A generalization of conjugacy in groups*, « Rendiconti del Seminario Matematico Università di Padova », *40*, 408–427 (1968).

[5] Tamaschke O., *A generalization of the second isomorphism theorem in group theory*. Accademia Nazionale dei Lincei, « Rendiconti della Classe di Scienze fisiche, matematiche e naturali », Serie VIII, *45*, 135–141 (1968).

[6] Tamaschke O., *On permutation groups*, « Annali di Matematica », Serie IV, *80*, 235–279 (1968).

[7] Tamaschke O., *On the theory of Schur–rings*, « Annali di Matematica », Serie IV, *81*, 1–43 (1969).

[8] Tamaschke O., *Permutationsstrukturen*. B. I. Hochschulskripten 710/710a, Bibliographisches Institut, Mannheim–Wien–Zürich 1969.

[9] Tamaschke O., *Schur–Ringe*. B. I. Hochschulskripten. Bibliographisches Institut, Mannheim – Wien – Zürich. To appear.