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A Cauchy problem for an integro-differential equation of parabolic type with a delayed argument

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Analisi matematica. — *A Cauchy problem for an integro-differential equation of parabolic type with a delayed argument* (*). Nota di MEHMET NAMIK OĞUZTÖRELI (**), presentata (***) al Socio M. PICONE.

RIASSUNTO. — In un precedente lavoro [1] noi investigammo un problema di Fourier della prima specie per una equazione integro-differenziale di tipo parabolico ad argomento ritardato. Nella presente Nota consideriamo un problema di Cauchy per la stessa equazione.

1. FORMULATION OF THE CAUCHY PROBLEM.

Let D be a domain in E^n, bounded by a smooth surface ∂D satisfying the conditions of Liapunov. Let t_0 be a fixed time instant, the initial instant, and h_i $(i = 0, 1, \cdots, m)$ be certain given numbers such that

$$0 = h_0 < h_1 < \cdots < h_m . \tag{1.1}$$

Let t_1 be a given number, such that $t_0 < t_1 < \infty$, and define the intervals

$$I_0 = [t_0 - h_m , t_0] , I_1 = (t_0 , t_1] , I_2 = [t_0 - h_m , t_1] . \tag{1.2}$$

Let $C_k^{1,2}$ $(k = 0, 1, 2)$ be the Banach spaces of real valued functions $u(t, x)$ continuously differentiable in t and twice continuously differentiable in x for $(t, x) \in I_k \times \bar{D}$ with norms

$$\| u \|_k^{(1,2)} = \sup_{(t,x) \in I_k \times \bar{D}} \left\{ |u| , \left| \frac{\partial u}{\partial t} \right| , \left| \frac{\partial u}{\partial x_i} \right| , \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right| \right\} \tag{1.3}$$

$$(1 \leq i , j \leq n , k = 0 , 1 , 2)$$

where $\bar{D}$ is the closure of D.

Further, let C_k^0 be the Banach space of all real valued functions $u(t, x)$ continuous on $I_k \times D$ with the norm

$$\| u \|_k^0 = \sup_{(t,x) \in I_k \times \bar{D}} | u | , \qquad (k = 0, 1, 2). \tag{1.4}$$

We define the operators A and B form $C_1^{1,2}$ into C_2^0 by the equations

$$A(u)(t, x) = \sum_{i=0}^{m} a_i(t, x) u(t - h_i , x) + \int_{t_0}^{t} K(t, \tau, x) u(\tau, x) d\tau \tag{1.5}$$

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and

$$B(u)(t,x) = \Delta_x u(t,x) + P_x u(t,x) + \int_D H(t,x,\eta)\, u(t,\eta)\, dV_\eta , \tag{1.6}$$

where dV_η is the volume element at $\eta \in D$ and

$$\Delta_x = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \quad , \quad P_x = \sum_{j=1}^{n} b_j(t,x) \frac{\partial}{\partial x_j} , \tag{1.7}$$

where

$a_i(t,x)$ and $b_j(t,x)$, $i = 0, 1, \cdots, m$ and $j = 1, \cdots, n$, are given functions belonging to $C_1^{1,2}$;

$K(t,\tau,x)$ is a given function continuously differentiable in t and τ, and twice continuously differentiable in x for $(t,\tau,x) \in I_1 \times I_1 \times \bar{D}$;

$H(t,x,y)$ is a given function continuously differentiable in t and twice continuously differentiable in x and y for $(t,x,y) \in I_1 \times D \times D$.

Now, consider the following linear functional differential equation:

$$\frac{\partial}{\partial t} u(t,x) = (A + B)(u)(t,x) + f(t,x) \tag{1.8}$$

for $(t,x) \in I_1 \times D$, where $f(t,x)$ is a given function belonging to $C_1^{1,2}$.

In this paper we investigate the following Cauchy problem:

Find a solution of Eq. (1.8) *for* $(t,x) \in I_1 \times D$ *subject to the initial condition*

$$u(t,x) = \varphi(t,x) \qquad \text{for} \quad (t,x) \in I_0 \times \bar{D}, \tag{1.9}$$

where $\varphi(t,x)$ *is a given function belonging to the space* $C_0^{1,2}$.

We show that this problem is well-posed. We construct the solution, and give representation formula for the solution.

2. Existence of a Solution.

Let $\Gamma(t,x;\tau,\xi)$ be a fundamental solution of the equation $Lu = 0$, where

$$L = \frac{\partial}{\partial t} - \Delta_x - P_x . \tag{2.1}$$

Let us write Eq. (1.8) in the form

$$L(u)(t,x) = f(t,x) + g(t,x,u) \tag{2.2}$$

for $(t,x) \in I_1 \times D$, where

$$g(t,x,u) = A(u)(t,x) + \int_D H(t,x,\eta)\, u(t,\eta)\, dV_\eta . \tag{2.3}$$

Hence, the solution $u(t, x)$ of our Cauchy problem statisfies the following linear integro-difference equation

$$u(t, x) = \int_{D} \Gamma(t, x; \tau, \xi)\, \varphi(t_0, \xi)\, dV_\xi + \tag{2.4}$$

$$+ \int_{t_0}^{t} \int_{D} \Gamma(t, x; \tau, \xi)\, g(\tau, \xi, u(\tau, \xi))\, dV_\xi\, d\tau$$

for $(t, x) \in I_1 \times D$. Thus, the solution of the Cauchy problem is equivalent to the solution of the integro-difference equation (2.4) subject to the initial condition (1.9).

Following the main lines, with certain natural modifications, of the proof in [1] for the existence of a solution of the Fourier problem of the first kind, we can easily show the existence of a solution of the Cauchy problem formulated above.

The uniqueness of the solution can be established by the help of a theorem due to H. Tanabe [3]-[5].

3. CONSTRUCTION OF THE SOLUTION. A REPRESENTATION FORMULA.

We now seek a solution to the Cauchy problem (1.8)-(1.9) in the following form:

$$u(t, x) = \int_{t_0}^{t} \int_{D} \tilde{L}_1(t, x; \tau, \xi)\, f(\tau, \xi)\, dV_\xi\, d\tau \tag{3.1}$$

$$+ \int_{I_0} \int_{D} \tilde{L}_2(t, x; \tau, \xi)\, \varphi(\tau, \xi)\, dV_\xi\, d\tau$$

where the kernels $\tilde{L}_1$ and $\tilde{L}_2$ are to be determined. By certain simple manipulations similar to those made in [1], we see that the function $\tilde{L}_1$ satisfies the integro-differential equation,

$$\frac{\partial \tilde{L}_1(t, x; \tau, \xi)}{\partial t} = \sum_{i=0}^{m} a_i(t, x)\, \tilde{L}_1(t - h_i, x; \tau, \xi) + \tag{3.2}$$

$$+ \int_{\tau}^{t} K(t, \sigma, x)\, \tilde{L}_1(\sigma, x; \tau, \xi)\, d\sigma + B(\tilde{L}_1)(t, x; \tau, \xi)$$

for $t_0 \leq \tau < t \leq t_1$ and $x, \xi \in D$, and the initial condition

$$\tilde{L}_1(t, x; \tau, \xi) = \begin{cases} 0 & \text{for } t < \tau, \\ \delta(x - \xi) & \text{for } t = \tau; \end{cases} \tag{3.3}$$

and the function $\tilde{L}_2(t, x; \tau, \xi)$ satisfies the integro-differential equation

$$\left(\frac{\partial}{\partial t} - (A + B)\right) \tilde{L}_2 = 0 \tag{3.4}$$

for $(t, x) \in I_1 \times D$, and the initial condition

$$\tilde{L}_2(t, x; \tau, \xi) = \delta(t - \tau)\, \delta(x - \xi) \tag{3.5}$$

for $(t, x) \in I_0 \times \bar{D}$, $(\tau, \xi) \in I_0 \times D$, where $\delta(t)$ and $\delta(x)$ are Dirac's functions. Thus, construction of the solution of our Cauchy problem is reduced to the construction of the kernel functions $\tilde{L}_1$ and $\tilde{L}_2$.

We now extend the definition of the fundamental solution $\Gamma(t, x; \tau, \xi)$, by setting

$$\Gamma(t, x; \tau, \xi) = 0 \qquad \text{for} \quad t < \tau \qquad \text{and} \quad x, \xi \in D. \tag{3.6}$$

Then we search a solution of Eq. (3.2) in the following form:

$$\tilde{L}_1(t, x; \tau, \xi) = \Gamma(t, x; \tau, \xi) + \int_\tau^t \int_D \Gamma(t, x; \sigma, \eta)\, \tilde{\mu}_1(\sigma, \eta)\, dV_\eta\, d\sigma, \tag{3.7}$$

where $\tilde{\mu}_1(t, x)$ is a function continuous on $I_1 \times \bar{D}$, except, perhaps for $t = \tau + h_i$ $(i = 1, \cdots, m)$ and $x = \xi$, which will be determined below: We assume that

$$\tilde{\mu}_1(t, x) = 0 \qquad \text{for} \quad t < \tau \qquad \text{and} \quad x \in \bar{D}. \tag{3.8}$$

Clearly, the function $\tilde{L}_1(t, x; \tau, \xi)$ defined by Eq. (3.7) satisfies the initial condition (3.3). To determine the $\tilde{\mu}_1$ so that the function $\tilde{L}_1(t, x; \tau, \xi)$ given by Eq. (3.7) will satisfy Eq. (3.2) for $t_0 \leq \tau < t \leq t_1$, $x \in D$, $x \neq \xi$, $\xi \in D$, we substitute it into Eq. (3.2). Applying the Dirichlet's formula we find the following singular Volterra equation of the second kind:

$$\tilde{\mu}_1(t, x) = \tilde{S}(t, x; \tau, \xi) + \int_\tau^t \int_D \tilde{S}(t, x; \sigma, \eta)\, \tilde{\mu}_1(\sigma, \eta)\, dV_\eta\, d\sigma, \tag{3.9}$$

where

$$\tilde{S}(t, x; \sigma, \eta) = \int_D H(t, x; \zeta)\, \Gamma(t, \zeta; \sigma, \eta)\, dV_\zeta + \tag{3.10}$$

$$+ \int_\sigma^t K(t, \theta, x)\, \Gamma(\theta, x; \sigma, \eta)\, d\theta + \sum_{i=1}^m a_i(t, x)\, \Gamma(t - h_i, x; \sigma, \eta).$$

Apparently, the function $\tilde{S}(t, x; \tau, \xi)$ has certain weak singularities occuring at $x = \xi$, and $t = \tau + h_i$ $(i = 1, \cdots, m)$, otherwise it is smooth. To solve the integral equation (3.9) we can use the method of successive approximations. Thus, we have

$$(3.11) \qquad \tilde{\mu}_1(t, x) = \tilde{S}(t, x; \tau, \xi) + \int_\tau^t \int_D \tilde{R}_1(t, x; \sigma, \eta)\, \tilde{S}(\sigma, \eta; \tau, \xi)\, dV_\eta\, d\sigma,$$

where $\tilde{R}_1$ is the resolvent of the kernel $\tilde{S}$. Clearly, Eqs. (3.7) and (3.11) uniquely determine the kernel $\tilde{L}_1(t, x; t, \xi)$.

We now seek a solution of Eq. (3.4) in the following form:

$$(3.12) \qquad \tilde{L}_2(t, x; \tau, \xi) = \begin{cases} \delta(t-\tau)\,\delta(x-\xi) & \text{for } t, \tau \in I_0 \text{ and } x, \xi \in \bar{D}, \\ \displaystyle\int_{t_0}^t \int_D \Gamma(t, x; \sigma, \eta)\, \tilde{\mu}_2(\sigma, \eta)\, dV_\eta\, d\sigma & \text{for } (t, x) \in I_1 \times \bar{D}, \end{cases}$$

where $\tilde{\mu}_2(t, x)$ is a function continuous on $I_1 \times D$, except perhaps at $t = t_0 + h_1, \cdots, t_0 + h_m$ and $x = \xi$. Clearly, the function $\tilde{L}_2(t, x; \tau, \xi)$ defined by Eq. (3.12) satisfies the initial condition (3.5) by virtue of Eq. (3.6).

To determine the function $\tilde{\mu}(t, x)$ so that the function $\tilde{L}_2(t, x; \tau, \xi)$ given by (3.12) will satisfy Eq. (3.4) for $(t, x) \in I_1 \times D$ for each $(\tau, \xi) \in I_0 \times \bar{D}$, we substitute it into Eq. (3.4). We then obtain the following singular Volterra integral equation of the second kind:

$$(3.13) \qquad \tilde{\mu}_2(t, x) = \tilde{E}(t, x; \tau, \xi) + \int_{t_0}^t \int_D \tilde{S}^*(t, x; \sigma, \eta)\, \tilde{\mu}_2(\sigma, \eta)\, dV_\eta\, d\sigma,$$

for $(t, x) \in I_1 \times D$, where

$$(3.14) \qquad \tilde{E}(t, x; \tau, \xi) = \begin{cases} \displaystyle\sum_{i=1}^m a_i(t, x)\, \delta(t - h_i - \tau)\, \delta(x - \xi) \\ \quad \text{for } t_0 < t < t_0 + h_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \displaystyle\sum_{i=k}^m a_i(t, x)\, \delta(t - h_i - \tau)\, \delta(x - \xi) \\ \quad \text{for } t_0 + h_{k-1} < t < t_0 + h_k, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \qquad t_0 + h_m < t \leq t_1, \end{cases}$$

and

$$(3.15) \qquad \tilde{S}^*(t, x; \sigma, \eta) = \int_D H(t, x, \zeta)\, \Gamma(t, \zeta; \sigma, \eta)\, dV_\zeta + \\ + \int_\sigma^t K(t, \theta, x)\, \Gamma(\theta, x; \sigma, \eta)\, d\theta + \tilde{F}(t, x; \sigma, \eta)$$

with

$$(3.16)\qquad \tilde{\tilde{F}}(t, x; \sigma, \eta) = \begin{cases} 0 \quad \text{for} \quad t_0 < t < t_0 + h_1, \\ \dots\dots\dots\dots\dots\dots \\ \sum_{i=1}^{k-1} a_i(t, x)\, \Gamma(t - h_i, x; \sigma, \eta) \\ \quad \text{for} \quad t_0 + h_{k-1} < t < t_0 + h_k, \\ \dots\dots\dots\dots\dots\dots \\ \sum_{i=1}^{m} a_i(t, x)\, \Gamma(t - h_i, x; \sigma, \eta) \\ \quad \text{for} \quad t_0 + h_m < t < t_1, \end{cases}$$

for $(\sigma, \eta) \in I_1 \times D$, $x \in D$. It is obvious that the functions $\tilde{\tilde{E}}$ and $\tilde{\tilde{F}}$ possess only weak singularities at $t = \sigma + t_0 + h_i$ $(i = 1, \cdots, m)$ and $x = \xi$.

Here again we can use the method of successive approximations to solve the singular integral equation (3.13). The solution is of the form

$$(3.17)\qquad \tilde{\tilde{\mu}}_2(t, x) = \tilde{\tilde{E}}(t, x; \tau, \xi) + \int_{t_0}^{t}\int_{D} \tilde{\tilde{R}}_2^*(t, x; \sigma, \eta)\, \tilde{\tilde{E}}(\sigma, \eta; \tau, \xi)\, dV_\eta\, d\sigma$$

where R_2^* is the resolvent of the kernel $\tilde{\tilde{S}}_2^*$.

Clearly, Eqs. (3.12) and (3.17) uniquely determine the kernel $\tilde{L}_2(t, x; \tau, \xi)$.

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