Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On quasi-normalizable operators II

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **46** (1969), n.5, p. 523–525.

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<http://www.bdim.eu/item?id=RLINA_1969_8_46_5_523_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1969.

Analisi matematica. — *On quasi–normalizable operators II.* Nota di VASILE ISTRĂŢESCU e IOANA ISTRĂŢESCU, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questa nota si danno alcuni risultati riguardanti gli operatori quasi–normalizzabili e cioè: un'applicazione riguardante una generalizzazione della forma quadratica di Hilbert e ancora un risultato riguardante lo spettro essenziale di una suddivisione della classe di operatori quasi–normalizzabili.

1. In the Note [3] a class of operators generalizing the class of symmetrizable operators was introduced. The purpose of the present Note is to give some applications.

For all definitions and results on quasi-normalizable operators we refer to [3].

2. W. Magnus has considered the following quadratic form

$$\sum_{n=m=1}' \frac{1}{(n+m)^{1-\alpha}} \frac{1}{(nm)^{\alpha/2}} x_n x_m \qquad 0 \leq \alpha \leq 1$$

and has proved that this is bounded if α is less than 1; for $\alpha = 1$ it is unbounded. For $\alpha = 0$ the above quadratic form is the Hilbert quadratic form.

Our generalization is as follows: suppose now that $\{\alpha_n\}$ is arithmetic progression of complex numbers with Re $\alpha > 0$ for every n and consider the quadratic form

$$(*) \qquad \sum_{n=m=0} \left(\frac{1}{\alpha_n + \bar{\alpha}_m}\right)^{1-\alpha} \frac{1}{(\alpha_n \alpha_m)^{\alpha/2}} x_n x_m \qquad 0 \leq \alpha < 1 .$$

THEOREM 1. *The quadratic form* (*) *is bounded.*

Proof. For the proof consider the Banach space of sequences $x = (x_0, x_1, \cdots, x_n, \cdots)$ with the norm

$$|x| = \sup |x_m| \, |\alpha_m|^s$$

when the parameter s stays between $\alpha/2$ and $1 - \alpha/2$.

Define an operator T by the relation

$$Tx = y = (y_0, y_1, \cdots)$$

where $y_n = \sum_0^\infty \frac{1}{(\alpha_n + \bar{\alpha}_m)^{1-\alpha}} \frac{1}{(\alpha_n \bar{\alpha}_m)^{\alpha/2}} x_m .$

(*) Nella seduta del 19 aprile 1969.

Since, it is easy to see that

$$|y_n| \, n^s \leq k \sum \left(\frac{1}{n+m}\right)^{1-\alpha} \frac{1}{(nm)^{\alpha/2}} \; \frac{n^s}{m^s} \; |x|$$

we see that T is bounded in the Banach norm $|\ |$. Since T is with the property $T^* T - TT^* = 0$ from theorem 1 of [3] it follows that T is bounded in the Hilbert norm and the theorem is proved.

Remark. We remark that the theorem is true under more general conditions i.e. for which the operator T is with the property

$$T^* T - TT^* \geq 0 .$$

3. In this section we give a result about the essential spectrum of an operator T which is hyponormalizable i.e. for which the relation (*) of [3] holds. Since there exists a great divergence in the literature over the definition of essential spectrum we use the following definition: $\sigma_e(T) = \sigma_e$ (the essential spectrum) in the set of all $\lambda \in \sigma(T)$ which are invariant under arbitrary compact perturbation, i.e.

$$\sigma_e(T) = \cap \, \sigma(T + K)$$

where the intersection is taken over all compact operators. Our result is

THEOREM 2. *Let* T *be an operator which is with property* (*) *of* [3]. *The essential spectrum of* T *are all* $\lambda \in \sigma(T)$ *which are not isolated eigenvalues of finite multiplicity.*

Proof. For the proof we use, as Coburn [1], J. Nieto [5] and others, the Schechter characterization of σ_e [6].

$$\sigma_e(T) = \{\lambda \, , T - \lambda \notin \mathfrak{F}\} \cup \{\lambda \, , (T - \lambda) \in \mathfrak{F} \, , \ \text{index}\,(T - \lambda) \neq 0\}$$

where $\mathfrak{F}$ denotes the class of Fredholm operators.

Also, for the proof we use the connection between spaces $B = X$ and its completion H under scalar product.

Suppose now that $\lambda \in \sigma(T)$ and index $(T - \lambda) = 0$ we can prove that this is possible if and only if λ is an isolated eigenvalue of finite multiplicity. Since if T is hyponormalizable then for all λ, $T - \lambda = T - \lambda I$ is also hyponormalizable, we may suppose without loss of generality that $\lambda = 0$. Thus the problem is: if T is a Fredholm operator of index zero which is not invertible if and only if zero is an isolated eigenvalue of finite multiplicity.

Suppose now that T is a Fredholm operator of index zero. Then R (T) is closed, N(T) is finite—dimensional subspace of X. But, from theorem 2.4 in [3], the index of T on the completion is the same, and thus, the index of T/H is zero. Since for λ small and $\neq 0$, $\lambda \in \rho(T/H)$, because $\lambda = 0$ is an isolated eigenvalue of finite multiplicity for T on H, combined with the fact that index is zero we have that such λ is in $\rho\,(T/X)$. This implies that zero is an isolated point in $\sigma\,(T/X)$.

For the converse, if zero is an isolated eigenvalue of finite multiplicity from theorem 2.4 in [3] it is also an isolated eigenvalue for T on H. The range of T is closed in H and thus we have a decomposition

$$H = R(T) \otimes N(T)$$

Since T on H is with G1-property, it is easy to see that

$$P_{H,X} = -\frac{1}{2\pi i} \int\limits_{\Gamma_\varepsilon = \{z, |z| = \varepsilon\}} (T - \lambda)^{-1} \, d\lambda$$

has the property $P_H x = 0$ for every $x \in H$ and since $P_X x = 0$ for $x \in X$ we obtain $R(P_X/X) \subset N(T/X)$ which, from assumptions, shows the range of P_X is finite dimensional. From theorem of Taylor, index of T is zero and the theorem is proved.

Remarks. In our proof we have used techniques and results given by A. Coburn and J. Nieto.

2. Using the above theorem it is easy to give an extension of a theorem of Nieto [5], theorem 2.

3. The theorem may be applied to some classes of operators generalizing the class of quasi-hermitian operators studied by J. Dieudonné, such there exists a bounded selfadjoint operator Q, $Q > 0$ such that QT is a hyponormal operator [4].

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