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Osvaldo Borghi, Ezio Marchi

On the product probability for an arbitrary family of spaces

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Matematica. — *On the product probability for an arbitrary family of spaces.* Nota di OSVALDO BORGHI ed EZIO MARCHI, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Alcune idee introdotte da C. Ionescu Tulcea [5] sulle probabilità negli spazi prodotti vengono qui utilizzate nella teoria generale dei giochi.

The infinite games given in a extensive form introduced by D. Gale and F. M. Stewart in [2], are based on the tree structure for the description of the internal states. A further approach by R. Aumann [1] makes no use of the rather cumbersome tree model. Indeed, the most natural way in which such evolution processes appear is concerned with bundle of paths of states. Then, one should deal with an overlapping of arbitrary families of probability spaces when mixed behaviour is presented. These suggest the idea of a connection with the treatment of probabilities in product spaces in the general sense considered by C. Ionescu Tulcea in [5]. Some results in this direction are presented here.

Let $\{(E_i, \mathcal{A}_i); i \in I)\}$ be any non-empty family of non-empty measurable spaces. E denotes the union of all the E_i. Following J. Neveu [4] we can construct the class $\mathcal{B} = \{B \subset E : B \cap E_i \in \mathcal{A}_i \text{ for all } i \in I\}$, which is a σ–field included in the class $\mathcal{S}(E)$ of all the subsets of E. Moreover, the condition $\mathcal{A}_i \cap E_j \subset \mathcal{A}_j$ for all $i, j \in I$ is equivalent to $E_i \in \mathcal{B}$ and $\mathcal{B} \cap E_i = \mathcal{A}_i$ for all $i \in I$. From now on, we assume it. For example, an arbitrary family of Borel sets in a Euclidean space endowed with the Borel σ–field, having the induced σ–field.

Let $\{(E_{i_1}, \mathcal{A}_{i_1}, P_{i_1}); i_1 \in I_1\}$ be an arbitrary family of probability spaces and let $(I_1, \mathcal{I}_1, Q_1)$ be a further probability space. For all $i_1 \in I_1$, $P_2^{i_1}$ denotes a transition probability defined on $E_{i_1} \times \mathcal{A}_2$, where $(E_2, \mathcal{A}_2)$ is a new measurable space. Given the mapping R which assigns $P_2^{i_1}(x_1, A_2)$ to the point $(i_1, x_1, A_2) \in \mathcal{G} \times A_2$ where $\mathcal{G}$ is the graph $\{(i_1, x_1) \in I_1 \times E_1 : \chi_{E_{i_1}}(x_1) = 1\}$ where here χ is the indicator. If S indicates the function that to each $i_1 \in I_1$ determines the value $P_{i_1}(B_1 \cap E_{i_1})$, then we have:

THEOREM. *If a): R is $(\mathcal{I}_1 \times \mathcal{B}_1) \cap \mathcal{G}$–measurable for all $A_2 \in \mathcal{A}_2$, where $\mathcal{G} \in \mathcal{I}_1 \times \mathcal{B}_1$, and b): S is $\mathcal{I}_1$–measurable for all $B_1 \in \mathcal{B}_1$, then there exists a unique probability P on $\mathcal{B}_1 \times \mathcal{A}_2$ such that for all measurable rectangles:*

$$P(B_1 \times A_2) = \int_{I_1} dQ_1(i_1) \int_{B_1 \cap E_{i_1}} dP_{i_1}(x_1)\, P_2^{i_1}(x_1, A_2).$$

(*) Nella seduta del 10 maggio 1969.

Proof: Consider for any $i_1 \in I_1$ the extended σ-field $\bar{\mathcal{A}}_{i_1} = \mathcal{A}_{i_1} \cup \mathcal{S}(E_1 - E_{i_1})$ and the probability $\bar{P}_{i_1}(\bar{A}_{i_1}) = P_{i_1}(A_{i_1})$ for all $\bar{A}_{i_1} \in \bar{\mathcal{A}}_{i_1}$. For all the measurable rectangle $B_1 \times A_2$ in the product σ-field $\mathcal{B}_1 \times \mathcal{A}_2$, we have for all $i_1 \in I_1$, the following relation

$$(1) \qquad P^{i_1}(B_1 \times A_2) = \int_{B_1} d\bar{P}_{i_1|\mathcal{B}_1}(x_1)\, \bar{R}(i_1, x_1, A_2) = \int_{B_1 \cap E_{i_1}} dP_{i_1}(x_1)\, R(i_1, x_1, A_2)$$

where $\bar{P}_{i_1|\mathcal{B}_1}$ is the restriction of $\bar{P}_{i_1}$ to the σ-field $\mathcal{B}_1$, and $\bar{R}$ equals R on $\mathcal{G}$ and $\bar{R}$ is zero outside $\mathcal{G}$.

The condition a) implies $\bar{R}$ is $\mathcal{J}_1 \times \mathcal{B}_1$-measurable. Note that the mapping $R(i_1, \cdot, A_2)$ is $\mathcal{A}_{i_1}$-measurable for all $i_1 \in I_1$, and for all $A_2 \in \mathcal{A}_2$.

The expression (1) determines an extension which is a transition probability on $I_1 \times (\mathcal{B}_1 \times \mathcal{A}_2)$, since $\bar{P}_{\cdot|\mathcal{B}_1}(B_1)$ is $\mathcal{J}_1$-measurable for all $B_1 \in \mathcal{B}_1$ implies that $P^{\cdot}(B_1 \times A_2)$ is a function $\mathcal{J}_1$-measurable for all the rectangles in $\mathcal{B}_1 \times \mathcal{A}_2$.

If we construct the monotone class $\mathcal{M}$ of all the $C \in \mathcal{B}_1 \times \mathcal{A}_2$ such that the function $P^{\cdot}(C)$ is $\mathcal{J}_1$-measurable and since it contains the generated field by the class of all the measurable rectangles, then the function $P^{\cdot}(C)$ results $\mathcal{J}_1$-measurable for all $C \in \mathcal{B}_1 \times \mathcal{A}_2$ (q.e.d.).

Conversely, the following proposition arises immediately. Let $(E_1, \mathcal{A}_1, P_1)$ be a probability space and $\{(E_{i_2}, \mathcal{A}_{i_2}); i_2 \in I_2\}$ and arbitrary family of measurable spaces and finally $(I_2, \mathcal{J}_2, Q_2)$ a further probability space. Moreover, for $i_2 \in I_2$, $P^1_{i_2}$ denotes a transition probability on $E_1 \times \mathcal{A}_{i_2}$. T indicates the function which assigns the value $P^1_{i_2}(x_1, B_2 \cap E_{i_2})$ to the point $(i_2, x_1, B_2) \in I_2 \times E_1 \times \mathcal{B}_2$.

PROPOSITION: *If a'*): T *is* $\mathcal{J}_2 \times \mathcal{A}_1$*-measurable for all* $B_2 \in \mathcal{A}_2$, *then there exists a unique probability* P *on* $\mathcal{A}_1 \times \mathcal{B}_2$ *such that for all measurable rectangles*:

$$P(A_1 \times B_2) = \int_{I_2} dQ_2(i_2) \int_{A_1} dP_1(x_1)\, P^1_{i_2}(x_1, B_2 \cap E_{i_2}).$$

Given an arbitrary family of probability spaces indexed by I_1 as in the first case and an arbitrary family of measurable spaces endowed by I_2 as in the proposition, one considers the transition probability $P^{i_1}_{i_2}$ defined on $E_{i_1} \times \mathcal{A}_{i_2}$ where i_1 is an element of the probability space $(I_1, \mathcal{J}_1, Q_1)$ and i_2 is a point belonging to $(I_2, \mathcal{J}_2)$, and finally Q^1_2 is a further transition probability on $I_1 \times \mathcal{J}_2$. Define the mapping U such that to the element $(i_1, x_1, i_2, B_2) \in \mathcal{G}_1 \times I_2 \times \mathcal{B}_2$ assigns the value $P^{i_1}_{i_2}(x_1, B_2 \cap E_{i_2})$.

COROLLARY. *If the condition b) of the theorem is valid and if a''): the function* U *is* $[(\mathcal{J}_1 \times \mathcal{B}_1) \cap \mathcal{G}_1] \times \mathcal{J}_2$*-measurable for all* $B_2 \in \mathcal{B}_2$, *where* $\mathcal{G}_1 \in \mathcal{J}_1 \times \mathcal{B}_1$, *then there exists a unique probability* P *on* $\mathcal{B}_1 \times \mathcal{B}_2$ *such that for all measurable rectangles*:

$$P(B_1 \times B_2) = \int_{I_1} dQ_1(i_1) \int_{I_2} Q^1_2(i_1, d_{i_2}) \int_{B_1 \cap E_{i_1}} dP_{i_1}(x_1)\, P^{i_1}_{i_2}(x_1, B_2 \cap E_{i_2}).$$

Proof: In fact the following expression

$$P_2^{i_1}(x_1, B_2) = \int_{I_2} Q_2^1(i_1, d_{i_2})\, P_{i_2}^{i_1}(x_1, B_2 \cap E_{i_2})$$

which holds true for all $(i_1, x_1) \in \mathcal{G}_1$ and all $B_2 \in \mathcal{B}_2$, define a transition probability on $E_{i_1} \times \mathcal{B}_2$ for all $i_1 \in I_1$. Then, the mapping $P_2^{\bullet}(\cdot, B_2)$ is $(\mathfrak{J}_1 \times \mathcal{B}_1) \cap \mathcal{G}_1$–measurable for all $B_2 \in \mathcal{B}_2$ (q.e.d.).

We will now generalize the previous results. Let $\{(E_{i_t}, \mathcal{A}_{i_t}) ; i_t \in I_t ; t \in N\}$ be a sequence of arbitrary families of measurable spaces. For any $t \in N$, $(I_t, \mathfrak{J}_t)$ is a measurable space, $P_{i_{t+1}}^{\omega_t}$ indicates a transition probability on $\prod_{j=0}^{t} E_{i_j} \times \mathcal{A}_{i_{t+1}}$ where $\omega_t = (i_0, \cdots, i_t)$, and $Q_{t+1}^{0\cdots t}$ is a transition probability on $\prod_{j=0}^{t} I_j \times \mathfrak{J}_{t+1}$ Moreover, for any $i_0 \in I_0$, P_{i_0} is a probability and Q_0 a probability on $\mathfrak{J}_0$. Given the function U which assigns to each point $(\omega_t, y_{t-1}, B_t) \in \prod_{j=0}^{t-1} \mathcal{G}_j \times I_t \times \mathcal{B}_t$ the value $P_{i_t}^{\omega_{t-1}}(y_{t-1}, B_t \cap E_{i_t})$, where $y_{t-1} = (x_0, \cdots, x_{t-1})$, then if Q denotes the probability on $\left(I = \prod_{t \geq 0} I_t, \mathfrak{J} = \prod_{t \geq 0} \mathfrak{J}_t\right)$ obtained from the $Q_{t+1}^{0\cdots t}$ whose restriction to $\prod_{j \leq t} \mathfrak{J}_j$ is Q_t by using the known theorem due to C. Ionescu Tulcea [5] we have the following general result:

THEOREM. *If the condition b) of the first theorem is satisfied, and if* α): *for all* $t \geq 1$, *the function* U *is* $\left[\left(\prod_{j=0}^{t-1} \mathfrak{J}_j \times \prod_{j=0}^{t-1} \mathcal{B}_j\right) \cap \prod_{j=0}^{t} \mathcal{G}_j\right] \times \mathfrak{J}_t$*–measurable, for all* $B_t \in \mathcal{B}_t$, *where* $\mathcal{G}_j \in \mathfrak{J}_j \times \mathcal{B}_j$ *for* $j = 0, \cdots, t-1$. *Then there exists a unique probability* P *on* $\mathcal{B} = \prod_{t \geq 0} \mathcal{B}_t$, *such that*

$$P(B) = \int_I dQ(\omega) \cdot P_\omega(B)$$

for all $B \in \mathcal{B}$, *where* P_ω, *with* $\omega \in I$ *is the unique probability on* $\mathcal{B}$ *such that the restriction to* $\prod_{j \leq t} \mathcal{B}_j$ *coincides with* P_{ω_t}. *This probability takes the following expression on the measurable rectangles*:

$$P_{\omega_t}\left(\prod_{j \leq t} B_j\right) = \int_{B_0 \cap E_{i_0}} dP_{i_0}(x_0) \int_{B_1 \cap E_{i_1}} P_{i_1}^{\omega_0}(y_0, dx_1) \cdots \int_{B_{t-1} \cap E_{i_{t-1}}} P_{i_{t-1}}^{\omega_{t-2}}(y_{t-2}, dx_{t-1}) \int_{B_t \cap E_{i_t}} P_{i_t}^{\omega_{t-1}}(y_{t-1}, dx_t).$$

Furthermore $P \Big| \prod_{j \leq t} \mathcal{B}_j = P_t$ *and*

$$P_t(C) = \int_{\prod_{j \leq t} I_j} dQ_t(\omega_t) \cdot P_{\omega_t}(C).$$

Proof: For any $t \geq 1$, the condition α) implies that for all $C \in \prod_{j \leq t} \mathcal{B}_j$, the mapping $\omega_t \to P_{\omega_t}(C)$ is $\prod_{j \leq t} \mathcal{I}_j$–measurable.

In order to prove it, one can take the recurrence formula

$$P_{\omega_t}\left(\prod_{j \leq t} B_j\right) = \int_{\prod_{j \leq t} B_j \cap \prod_{j \leq t-1} E_j} dP_{\omega_{t-1}}(y_{t-1})\, P_{i_t}^{\omega_{t-1}}(y_{t-1}, B_t \cap E_{i_t})$$

and by induction, the validity for the measurable rectangles holds. Then, this is true for all $C \in \prod_{j \leq t} \mathcal{B}_j$.

Given $\omega \in I$, in a similar way like in the case just mentioned for Q, a unique probability on $\prod_t \mathcal{B}_j \cap E_{i_t}$, where $\omega = (i_0, \cdots, i_t, \cdots)$, can be obtained. Since

$$\prod_t \mathcal{B}_t = \left[\prod_t \mathcal{B}_t \cap (E_t - E_{i_t})\right] \cup \left[\prod_t \mathcal{B}_t \cap E_{i_t}\right],$$

there exists a unique probability P_ω on $\mathcal{B}$ such that $P_\omega \Big| \prod_{j \leq t} \mathcal{B}_j = P_{\omega_t}$.

The σ–field generated by the Boolean algebra $\bigcup_n \prod_{j \leq n} \mathcal{B}_j$ coincides with $\mathcal{B}$. Hence, $P.(B)$ is $\mathcal{I}$–measurable for all $B \in \mathcal{B}$.

The unicity is simple.

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