Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Aubrey William Ingleton

Tangent flag bundles and generalised Jacobian varieties. Nota II

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **46** (1969), n.5, p. 505–510.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1969_8_46_5_505_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1969.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 10 maggio 1969

Presiede il Presidente Beniamino Segre

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Tangent flag bundles and generalized Jacobian varieties.* Nota II di Aubrey William Ingleton, presentata (*) dal Socio B. Segre.

Riassunto. — Ved. la Nota I a p. 323 di questi «Rendiconti».

Linear Systems and Jacobians.

2.0. *"Ehresmann" subvarieties* (1) *of* V^Δ. Let

$$\mathfrak{L} : \mathfrak{L}_1 \subset \mathfrak{L}_2 \subset \cdots \subset \mathfrak{L}_t \quad , \quad \dim \mathfrak{L}_i = i - 1,$$

be a nest of linear systems of primals on V. For each $q = 0, \cdots, d$ and any given tangent flag S to V, let $\mathfrak{L}_i(q, S)$ denote the linear system consisting of those members of $\mathfrak{L}_i$ to which S_q is (formally) tangent at S_0 (in particular all the members through S_0 when $q = 0$, those with a singularity at S_0 when $q = d$). Then, corresponding to any (h, t)–index $\boldsymbol{k}$, $0 \leq h \leq d$, we define $[\boldsymbol{k} ; \mathfrak{L} \mid V^\Delta]$ to be the subvariety of V^Δ consisting of all the flags satisfying the conditions

$$\dim \mathfrak{L}_i(q, S) \geq d_i(q ; \boldsymbol{k}) - 1 \qquad (i = 1, \cdots, t ; q \in Q_i(\boldsymbol{k})) \tag{2.0.1}$$

(notation as in 1.2). The cohomology class dual to $[\boldsymbol{k} ; \mathfrak{L} \mid V^\Delta]$ will be denoted by $[\boldsymbol{k} ; \mathfrak{L} \mid V^\Delta]^*$. We then define $\omega_{r,s}(q, \mathfrak{L} \mid V^\Delta]$, $\omega(q, \mathfrak{L} \mid V^\Delta)$ by analogy with (1.2.2), (1.2.3).

(*) Nella seduta del 19 aprile 1969.

(1) These varieties generalize the «lifts» of linear systems to the tangent direction bundle $V^* = T(1; V)$ considered in [3] and [10].

If V is in general position relative to the fixed flag E of 1.2, then, for the nest $\mathfrak{L}$ where $\mathfrak{L}_i$ is cut on V by the primes through E_{n-i}, we have

$$[\boldsymbol{k} : \mathfrak{L} \mid V^{\Delta}]^* = \theta^* [\boldsymbol{k} \mid W]^*,$$

where θ is the injection introduced in 1.5.

2.1. *Definition of Jacobians*. An *indexed family* of nests of linear systems comprises nests

$$\mathfrak{L}^{(\alpha)} : \mathfrak{L}_1^{(\alpha)} \subset \cdots \subset \mathfrak{L}_{t_\alpha}^{(\alpha)} \qquad (\alpha = 1, \cdots, u)$$

together with, for each α, an (h_α, t_α)–index $\boldsymbol{k}^{(\alpha)}$, $0 \leq h_\alpha \leq d$. The *Jacobian* of the indexed family is defined to be the locus

$$J = J(\boldsymbol{k}^{(1)}; \cdots; \boldsymbol{k}^{(u)}; \mathfrak{L}^{(1)}, \cdots, \mathfrak{L}^{(u)})$$

of points on V which are the origins S_0 of tangent flags S to V satisfying all the conditions

$$\dim \mathfrak{L}_i^{(\alpha)}(q, S) \geq d_i(q; \boldsymbol{k}^{(\alpha)}) - 1$$

$$(\alpha = 1, \cdots, u; i = 1, \cdots, t_\alpha; q \in Q_i(\boldsymbol{k}^{(\alpha)})).$$

That is to say, J is the projection on V of the subvariety

$$J^{\Delta} = \bigcap_{\alpha=1}^{u} [\boldsymbol{k}^{(\alpha)}; \mathfrak{L}^{(\alpha)} \mid V^{\Delta}]$$

of V^{Δ}.

We shall assume that the indexed family is sufficiently general for each component of J^{Δ} to have the correct dimension and to occur with multiplicity one in the intersection of the $[\boldsymbol{k}^{(\alpha)}; \mathfrak{L}^{(\alpha)} \mid V^{\Delta}]$. The cohomology class j^{Δ} dual to J^{Δ} is then given by

$$(2.1.1) \qquad j^{\Delta} = \prod_{\alpha=1}^{u} [\boldsymbol{k}^{(\alpha)}; \mathfrak{L}^{(\alpha)} \mid V^{\Delta}]^*.$$

Because of (1.5.1), any element of $H^*(V^{\Delta})$ can be expressed as a (unique) polynomial in $\delta_1, \cdots, \delta^{d-1}$ of degree $\leq d - i$ in δ_i and with coefficients in $\rho^* H^*(V)$. Suppose then that

$$(2.1.2) \qquad j^{\Delta} = \sum_{\substack{0 \leq \lambda_i \leq d-i \\ (i=1,\cdots,d-1)}} j^{\Delta}_{\lambda_1 \cdots \lambda_{d-1}} \delta_1^{\lambda_1} \cdots \delta_{d-1}^{\lambda_{d-1}}.$$

Let

$$(2.1.3) \qquad \begin{cases} Q(J) = \bigcup_{\alpha=1}^{u} Q(\boldsymbol{k}^{(\alpha)}), \\ Q(J) \cup \{0, d\} = \{q_0, \cdots, q_m\}, q_0 < q_1 < \cdots < q_m. \end{cases}$$

Then $j^{\Delta} \in \pi^* H^*(T(q_1, \cdots, q_m; V))$ and so $j^{\Delta}_{\lambda_1 \cdots \lambda_{d-1}} = 0$ unless, for all $i \notin Q(J)$, $\lambda_i = \lambda_{i+1}$ $(= 0$ if $i = d - 1)$. Thus the highest degree term which can appear in (2.1.2) with non-vanishing coefficient is of degree $d - q_h$ in δ_i for $q_{h-1} < i \leq q_h$. We shall refer to the coefficient of this term simply as the *leading coefficient.*

(2.1.4) LEMMA. *If* $j \in \mathrm{H}^*(\mathrm{V})$ *is dual to the Jacobian* J *then* $\rho^* j$ *is the leading coefficient in* (2.1.2).

2.2. Since ρ^* is a monomorphism the cohomology class j is determined once we know the expression (2.1.2) for j^Δ. The first step is to express the $[\boldsymbol{k}^{(\alpha)} ; \mathfrak{L}^{(\alpha)} \mid \mathrm{V}^\Delta]^*$ as polynomials in $\delta_1, \cdots, \delta_d$— this step we consider later; the second, since the polynomial product given directly by (2.1.1) will in general be of too high degree, is to effect the reduction to the standard form (2.1.2) using (1.5.1). This step is pure algebra and it is possible to find an explicit formula for the leading coefficient in the resulting standard form. (Cfr. [3], proof of Theorem 5.2, for the case $\mathrm{Q}(\mathrm{J}) = \{1\}$). Combining this with (2.1.4) we obtain.

(2.2.1) THEOREM. *Let* $\mathrm{Q}(\mathrm{J}), q_0, \cdots, q_m$ *be defined as in* (2.1.3) *and let* q *be the greatest integer in* $\mathrm{Q}(\mathrm{J})$. *Suppose that* $j^\Delta = \rho^* \mathrm{P}(\delta_1, \cdots, \delta_q)$ (ρ^* applied only to the coefficients—not to the δ_i), *where* $\mathrm{P}(x_1, \cdots, x_q)$ *is a polynomial of degree* $\leq r$ *in each* x_i *with coefficients in* $\mathrm{H}^*(\mathrm{V})$, *and define a "reversed polynomial"* $\hat{\mathrm{P}}$ *by*

$$\hat{\mathrm{P}}(x_1, \cdots, x_q) = (x_1 x_2 \cdots x_q)^r\, \mathrm{P}\left(\frac{1}{x_1}, \cdots, \frac{1}{x_q}\right).$$

Let

$$\tilde{\mathrm{C}}(x) = (1 + c_1(\mathrm{V})\, x + \cdots + c_d(\mathrm{V})\, x^d)^{-1}.$$

Then the cohomology class of the Jacobian J *is equal to the coefficient of* $x_1^{q+r-1} x_2^{q+r-2} \cdots x_q^r$ *in*

$$\prod_{h=1}^{q} \tilde{\mathrm{C}}(x_h) \prod_{i=h+1}^{q} (x_h - x_i) \prod_{l=1}^{m-1} (x_{q_{l-1}+1} \cdots x_{q_l})^{d-q_l}\, \hat{\mathrm{P}}(x_1, \cdots, x_q).$$

2.3. THEOREM. (*The intersection formula*). *For a sufficiently general nest of linear systems*

$$\mathfrak{L} : \mathfrak{L}_1 \subset \mathfrak{L}_2 \subset \cdots \subset \mathfrak{L}_t,$$

we have, for any $(q, t-1)$*-index* $\boldsymbol{k}$, $0 \leq q \leq d$,

(2.3.1) $\omega(q, \mathfrak{L} \mid \mathrm{V}^\Delta)\, [\boldsymbol{k} ; \mathfrak{L} \mid \mathrm{V}^\Delta]^* = \Sigma\, [k_0, \cdots, k_{i-1}, k'_i, k_{i+1}, \cdots, k_q ; \mathfrak{L} \mid \mathrm{V}^\Delta]^*,$

where, for each $i = 0, \cdots, q$, k'_i *is the smallest integer* $> k_i$ *which is not in* $\{k_0, \cdots, k_q\}$, *and the summation is over all* i *such that there is no* h, $i < h \leq q$, *for which* $k_i < k_h < k'_i$.

2.4. In the special case when $\mathrm{V} = \mathrm{P}_n(\mathbf{C})$, $\mathrm{V}^\Delta = \mathrm{F}(n+1)$, and $\mathfrak{L}_i$ consists of the primes through E_{n-i} ($i = 1, \cdots, n$), we have $[\boldsymbol{k} ; \mathfrak{L} \mid \mathrm{V}^\Delta]^* = [\boldsymbol{k} ; \mathrm{F}]^*$ and (2.3.1) reduces to an intersection formula (2.3.1′) in $\mathrm{H}^*(\mathrm{F})$ which is included in Monk's ([8] Theorem 3). Since (2.3.1′) is already sufficient to express any $[\boldsymbol{k}\,;\,\mathrm{F}]^*$ as a polynomial in the $\omega(q\,;\,\mathrm{F})$, it follows that any relation which holds between Ehresmann classes independently of n must be an algebraic consequence of (2.3.1′) and so imply a formally similar relation between the classes $[\boldsymbol{k}\,;\,\mathfrak{L}\mid \mathrm{V}^{\Delta}]^*$ for any given V and $\mathfrak{L}$. In particular this remark applies to the cases of Monk's intersection formula which are not included in (2.3.1′) and to the Grassmannian formula (1.4.2).

To complete the picture we need the expressions for the $\omega(q\,,\mathfrak{L}\mid \mathrm{V}^{\Delta})$ in terms of the δ_i. These are given by

(2.4.1) LEMMA. *For a sufficiently general nest of linear systems*

$$\mathfrak{L} : \mathfrak{L}_1 \subset \mathfrak{L}_2 \subset \cdots \subset \mathfrak{L}_t\,,$$

and for a positive integer $q \leq \min(t-1\,,d)$,

$$\begin{aligned}\omega(q\,,\mathfrak{L}\mid \mathrm{V}^{\Delta}) &= (q+1)\,\omega(0\,,\mathfrak{L}\mid \mathrm{V}^{\Delta}) + \delta_1 + \cdots + \delta_q \\ &= (q+1)\,(\rho^* a) + \delta_1 + \cdots + \delta_q\,,\end{aligned}$$

where $a \in \mathrm{H}^2(\mathrm{V})$ *is the cohomology class dual to the (unique) member of* $\mathfrak{L}_1$.

2.5. THEOREM. (*The invariance principle*). *For any given proper* $(q\,,t)$*-index* $\boldsymbol{k}$ *there is a polynomial* $\mathrm{J}_{\boldsymbol{k}}(x_0\,,\cdots,x_q)$ *(depending only on* $\boldsymbol{k}$*) with integral coefficients such that, for any non-singular variety* V *of dimension* $\geq q$ *and any sufficiently general nest* $\mathfrak{L}$ *of linear systems on* V *with top dimension* $\geq t-1$,

$$[\boldsymbol{k}\,;\,\mathfrak{L}\mid \mathrm{V}^{\Delta}]^* = \mathrm{J}_{\boldsymbol{k}}(\rho^* a\,,\delta_1\,,\cdots,\delta_q)\,, \tag{2.5.1}$$

where $a \in \mathrm{H}^2(\mathrm{V})$ *is dual to the member of* $\mathfrak{L}_1$, *and, for any flag manifold* $\mathrm{F} = \mathrm{F}(n+1)$ *with* $n \geq t$,

$$[\boldsymbol{k}\,;\,\mathrm{F}]^* = \mathrm{J}_{\boldsymbol{k}}(-\gamma_0\,,\gamma_0-\gamma_1\,,\cdots,\gamma_0-\gamma_q). \tag{2.5.2}$$

The proof of (2.5.1), and the determination in general of a polynomial $\mathrm{J}_{\boldsymbol{k}}$, depends on repeated use of the intersection formula (2.3.1) to identify $[\boldsymbol{k}\,;\,\mathfrak{L}\mid \mathrm{V}^{\Delta}]^*$ with a polynomial in the $\omega(q\,,\mathfrak{L}\mid \mathrm{V}^{\Delta})$, in a form which is clearly independent of $\mathfrak{L}$ and V, and then applying (2.4.1). The flag-manifold interpretation (2.5.2) is obtained from the identification of $\mathrm{F}(n+1)$ with $\mathrm{P}_n(\mathbf{C})^{\Delta}$ (see 2.4) and means that the calculus of Ehresmann subvarieties on V^{Δ}, and hence of Jacobians, is completely determined by a knowledge of the corresponding calculus on $\mathrm{F}(n+1)$.

2.6. In particular, it is now possible, combining (1.4.3), (2.1.1), (2.5.1) and (2.5.2), to write down explicitly the polynomial $\mathrm{P}(x_1\,,\cdots,x_q)$ of (2.2.1) for the Jacobian of any indexed family with

$$k_0^{(\alpha)} < k_1^{(\alpha)} < \cdots < k_{h_\alpha}^{(\alpha)} \qquad (\alpha = 1\,,\cdots,u)$$

(i.e. each nest imposing conditions on only one flag-component), but no useful purpose would be served by actually exhibiting the polynomial here.

Instead we conclude by restricting our attention still further to indices of the type

$$\boldsymbol{k} = (0, r+1, r+2, \cdots, r+p), \ 0 < p < d.$$

The corresponding subvariety $[\boldsymbol{k} ; \mathfrak{L} \mid V^{\Delta}]$ consists of the tangent flags satisfying the single condition

$$\dim \mathfrak{L}_{r+1}(p, S) \geq r - 1.$$

Thus only one linear system of the nest ($\mathfrak{L}_{r+1}$ of dimension r) is involved and, unless S_0 is a base point, the condition requires simply that S_p be tangent at S_0 to every member of $\mathfrak{L}_{r+1}$ through S_0. From (2.5.2) and (1.4.4) we see that the corresponding polynomial $J_{\boldsymbol{k}}(x_0, \cdots, x_p)$ of 2.5 is given by

$$\begin{aligned} J_k(-\gamma_0, \gamma_0 - \gamma_1, \cdots, \gamma_0 - \gamma_p) &= [\boldsymbol{k} ; F]^* \\ &= \omega_{p,r}(p ; F) \\ &= (-1)^{pr} \hat{\bar{\sigma}}_r(\gamma_0, \cdots, \gamma_p). \end{aligned}$$

Thus

$$J_{\boldsymbol{k}}(x_0, \cdots, x_q) = \hat{\bar{\sigma}}_r(x_0, x_0 + x_1, \cdots, x_0 + x_q)$$

and so, from (2.5.1),

$$\begin{aligned} [\boldsymbol{k} ; \mathfrak{L} \mid V^{\Delta}]^* &= \omega_{p,r}(p, \mathfrak{L} \mid V^{\Delta}) \\ &= \hat{\bar{\sigma}}_r(\rho^* a, \delta_1 + \rho^* a, \cdots, \delta_p + \rho^* a). \end{aligned} \tag{2.6.1}$$

Now suppose that we are given a (sufficiently general) family of linear systems (*not* nests) $\mathfrak{L}^{(\alpha)}$, $\dim \mathfrak{L}^{(\alpha)} = r_\alpha$, and integers p_α, $0 < p_\alpha < d$, $(\alpha = 1, \cdots, u)$. We define the $(p_1, \cdots, p_u)$–Jacobian of $\mathfrak{L}^{(1)}, \cdots, \mathfrak{L}^{(u)}$ to be the locus of origins S_0 of tangent flags S satisfying the conditions

$$\dim \mathfrak{L}^{(\alpha)}(p_\alpha, S) \geq r_\alpha - 1 \qquad (\alpha = 1, \cdots, u).$$

Using (2.1.1) and (2.6.1) we see that the cohomology class of such a Jacobian is given by (2.2.1) with $q = \max p_\alpha$, $r = \max r_\alpha$ and

$$P(x_1, \cdots, x_q) = \prod_{\alpha=1}^{u} \hat{\bar{\sigma}}_{r_\alpha}(a_\alpha, x_1 + a_\alpha, \cdots, x_{p_\alpha} + a_\alpha),$$

where $a_\alpha \in H^2(V)$ is dual to a general member of $\mathfrak{L}^{(\alpha)}$. This is the simple case referred to in the introduction. The $(p, p, \cdots, p)$–Jacobian with $\Sigma r_\alpha = d - p + 1$ is classical; the $(1, 1, \cdots, 1)$–Jacobian with no restriction on Σr_α was considered in [3].

References.

[1] F. Severi, *Fondamenti per la geometria sulle varietà algebriche*, « Ann. Mat. Pura Appl. », (4) *32*, 1–81 (1951).

[2] D. Monk, *Jacobians of linear systems on an algebraic variety*, « Proc. Camb. Phil. Soc. », *52*, 198–201 (1956).

[3] A. W. Ingleton and D. B. Scott, *The tangent direction bundle of an algebraic variety and generalized Jacobians of linear systems*, « Ann. Mat. Pura Appl. », (4) *56*, 359–373 (1961).

[4] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, « Ann. Math. », (2) *57*, 115–207 (1953).

[5] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces I*; *II*, « Amer. J. Math. », *80*, 458–538 (1958); *81*, 315–382 (1959).

[6] F. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie* (Ergebnisse 1956).

[7] C. Ehresmann, *Sur la topologie de certains espaces homogènes*, « Ann. Math. », (2) *35*, 396–443 (1934).

[8] D. Monk, *The geometry of flag manifolds*, « Proc. London Math. Soc. », (3) *9*, 253–286 (1959).

[9] W. V. D. Hodge and D. Pedoe, *Methods of algebraic geometry*, Vol. II (Cambridge 1952).

[10] D. B. Scott, *Tangent-direction bundles of algebraic varieties*, « Proc. London Math. Soc. », (3) *11*, 57–79 (1961).