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A Characterization of Equilibrium Points of Bimatrix Games

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Ricerca operativa. — *A Characterization of Equilibrium Points of Bimatrix Games*. Nota di ADI BEN–ISRAEL e MICHAEL J. L. KIRBY, presentata dal Socio B. SEGRE.

RIASSUNTO. — I punti di equilibrio dei giochi bimatriciali vengon qui caratterizzati da certe sottomatrici della matrice di retribuzione.

NOTATIONS AND PRELIMINARIES. - We denote by:

R^n = the n-dimensional real vector space,
$R^n_+ = \{x \in R^n : x \geqq 0\}$ the nonnegative orthant in R^n,
e = the vector whose components are all 1 and whose dimension is to be determined by the context,
$M = \{1, 2, \cdots, m\}$,
$N = \{1, 2, \cdots, n\}$.

For any subsets $I \subset M$, $J \subset N$, let

$$\bar{I} = \{i : i \in M , i \notin I\} \quad , \quad \bar{J} = \{j : j \in N , j \notin J\} .$$

For any $m \times n$ real matrix

$$A = (a_{ij}) \qquad i \in M , j \in N,$$

define the submatrix

$$A(I/J) = (a_{ij}) \qquad i \in I , j \in J .$$

For any vector $x = (x_i) \in R^m$, define the subvector

$$x(I) = (x_i) , \qquad i \in I .$$

For any subspace L of R^n, P_L denotes the perpendicular projection on L.
For any $m \times n$ real matrix A denote by

R(A) the range space of A,
N(A) the null space of A,
A^T the transpose of A,
A^+ the generalized inverse of A, [8].

We recall from [2] that

$$x = A^+ b + N(A)$$

is the general solution of

$$Ax = b$$

whenever solvable; also

$$A^{+T} = A^{T+} \qquad \text{and} \qquad AA^+ = P_{R(A)} .$$

(*) Nella seduta del 19 aprile 1969.

A *bimatrix game* is a two person game defined by two $m \times n$ matrices $A = (a_{ij})$, $B = (b_{ij})$ so that if player I chooses $i \in M$ and player II chooses $j \in N$ then the payoffs are a_{ij} to player I and b_{ij} to player II. Without loss of generality the matrices A, B can be taken to be positive matrices, [5]. A point $(x, y) \in R^{m+n}$ is an *equilibrium point* of the bimatrix game (A, B) if and only if (x, y) satisfies

$$x \in R^m_+ \quad , \quad e^T x = 1 , \tag{1 a}$$

$$y \in R^n_+ \quad , \quad e^T y = 1 , \tag{1 b}$$

$$B^T x \leqq (x^T B y) e , \tag{2 a}$$

$$A y \leqq (x^T A y) e . \tag{2 b}$$

Equilibrium points of bimatrix games, whose existence was first proved by Nash [7], were further studied in [3], [4], [5] and [6].

(1) and (2) imply the following complementary slackness conditions

$$x^T [A y - (x^T A y) e] = 0 , \tag{3 a}$$

$$y^T [B^T x - (x^T B y) e] = 0 . \tag{3 b}$$

RESULTS. – Equilibrium points of a bimatrix game are characterized in the following

THEOREM:

Assumptions: Let A, B be positive $m \times n$ matrices and let $x \in R^m_+$, $y \in R^n_+$.

Conclusions: (x, y) is an equilibrium point for the bimatrix game (A, B) if and only if there are sets $I \subset M$, $J \subset N$ for which the following 8 conditions are satisfied.

$$e = A(I/J)\, u \qquad \text{for some } u \geqq 0, \tag{4}$$

$$e = B(I/J)^T v \qquad \text{for some } v \geqq 0, \tag{5}$$

$$x(\bar{I}) = 0 , \tag{6}$$

$$y(\bar{J}) = 0 , \tag{7}$$

$$x(I) = B(I/J)^{T+} e \left[\frac{1 - e^T w}{e^T B(I/J)^+ e} \right] + w, \qquad \text{for some } w \in N(B(I/J)^T), \tag{8}$$

$$y(J) = A(I/J)^+ e \left[\frac{1 - e^T z}{e^T A(I/J)^+ e} \right] + z , \qquad \text{for some } z \in N(A(I/J)), \tag{9}$$

$$B(I/N)^T x(I) \leqq \left[\frac{1 - e^T w}{e^T B(I/J)^+ e} \right] e , \tag{10}$$

$$A(M/J)\, y(J) \leqq \left[\frac{1 - e^T z}{e^T A(I/J)^+ e} \right] e . \tag{11}$$

Proof. – 1. *If*:

First we show that for the z defined by (9), $e^T z \neq 1$. For if $e^T z = 1$, then (11) implies that $y(J) = 0$ (since A is positive), so by (9) $z = 0$ contradicting $e^T z = 1$.

Now we show that y, satisfying (7) and (9), is a probability vector:

$$\begin{aligned} e^T y &= e^T y(J) && \text{by (7)} \\ &= 1 && \text{by (9)}. \end{aligned}$$

Hence

(1 *b*) is satisfied.

(1 *a*) is similarly proved.

To prove (2 *b*) we calculate

$$\begin{aligned} x^T A y &= x(I)^T A(I/J)\, y(J) && \text{by (6), (7)} \\ &= x(I)^T A(I/J)\, A(I/J)^+ e \left[\frac{1 - e^T z}{e^T A(I/J)^+ e}\right] + x(I)^T A(I/J) z && \text{by (9)} \\ &= x(I)^T e \left[\frac{1 - e^T z}{e^T A(I/J)^+ e}\right] && \text{since } z \in N(A(I/J)),\ (4) \text{ and } \\ & && A(I/J)\, A(I/J)^+ = P_{R(A(I/J))} \\ &= \frac{1 - e^T z}{e^T A(I/J)^+ e} && \text{by } (1a) \text{ and } (6) \end{aligned} \tag{12}$$

(2 *b*) follows now from (11) and (12)

(2 *a*) is similarly proved.

2. *Only if*:

Let (x, y) be an equilibrium point and define

$$I = \left\{ i : \sum_{j=1}^{n} a_{ij} y_j = x^T A y \right\} \tag{13}$$

$$J = \left\{ j : \sum_{i=1}^{m} b_{ij} x_i = x^T B y \right\}. \tag{14}$$

Both I, J are nonempty since I empty implies that

$$Ay < (x^T Ay)\, e$$

which implies by using (1 *a*) that

$$x^T A y < x^T A y, \quad \text{a contradiction.}$$

(6) and (7) follow from (2) (3) and the definitions (13) (14).

Using (7) and (13) we get

$$A(I/J)\, y(J) = (x^T A y)\, e \tag{15}$$

which proves (4) since $y(J) \geqq 0$, $x^T A y > 0$. (5) is similarly proved. The general solution of (15) is

$$y(z) = A(I/J)^+ e\,(x^T A y) + z\,, \qquad \text{for some} \quad z \in N(A(I/J))\,. \tag{16}$$

But

$$\begin{aligned} 1 &= e^T y \\ &= e^T y(J)\,, && \text{by (7)} \\ &= e^T A(I/J)^+ e\,(x^T A y) + e^T z\,. \end{aligned}$$

In general, $A(I/J)$ positive does not imply that $e^T A(I/J)^+ e \neq 0$. For example

$$A(I/J) = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} \;, \quad A(I/J)^+ = \frac{1}{2} \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}.$$

Since a positive constant may be added to all elements of A and B without changing the equilibrium points, it can thus be assumed without loss of generality that

$$e^T A(I/J)^+ e \neq 0\,, \qquad \text{by (16)}$$

so

$$x^T A y = \frac{1 - e^T z}{e^T A(I/J)^+ e}$$

which when inserted in (16) and (2 *b*) gives (9) and (11) respectively. (8) and (10) are similarly proved. *Q.E.D.*

Remarks.

1. An important special class of bimatrix games are 2-person 0-sum games, in which case

$$B = -A\,. \tag{17}$$

Then $e^T w = 0$ in (8) and $e^T z = 0$ in (9), since in this case

$$\begin{aligned} & e \in R(A(I/J)) && \text{by (7),} \\ & e \in R(A(I/J)^T) && \text{by (5) and (17),} \\ & w \in N(A(I/J)^T) && \text{by (8) and (17),} \\ & z \in N(A(I/J)) && \text{by (9).} \end{aligned}$$

We use here the fact that for any matrix A, $R(A)$ and $N(A^T)$ are orthogonal subspaces.

For 2-person 0-sum games the theorem of [1] is thus a corollary of our theorem. In particular, the value of the game is then $\dfrac{1}{e^T A(I/J)^+ e}$

2. In general $e^T w$ and $e^T z$ are not zero, as shown by the following example. Let

$$A = \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

then $(x, y) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix} \right)$ is an equilibrium point for all $0 \leq \alpha \leq 1$. For $0 < \alpha < 1$ (13) and (14) give

$$I = \{1\} \quad , \quad J = \{1, 2\}$$

and the submatrices $A(I/J)$, $B(I/J)$ are

$$A(I/J) = (1 \;\; 2) \quad , \quad B(I/J) = (1 \;\; 1)$$

with

$$A(I/J)^+ = \frac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad , \quad B(I/J)^+ = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Now z defined by (9) is of the form

$$z = \begin{pmatrix} 2\beta \\ -\beta \end{pmatrix} \text{ for some real } \beta, \quad \text{since} \quad z \in N(A(I/J)).$$

But (9) gives:

$$y(J) = \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix} = A(I/J)^+ e \left[\frac{1 - e^T z}{e^T A(I/J)^+ e} \right] + z$$

$$= \frac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \frac{1-\beta}{\left(\frac{3}{5}\right)} + \begin{pmatrix} 2\beta \\ -\beta \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} + \frac{5}{3}\beta \\ \frac{2}{3} - \frac{5}{3}\beta \end{pmatrix},$$

so that $\beta = \frac{3}{5}\alpha - \frac{1}{5}$, and $\beta \neq 0$ if $\alpha \neq \frac{1}{3}$ in which case $e^T z = \beta \neq 0$. In this example, however, $w = 0$ since

$$N(B(I/J)^T) = \{0\}.$$

The payoffs to the players at the equilibrium point $\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix} \right)$ are thus

$$x^T A(I/J) y = \frac{1 - e^T z}{e^T A(I/J)^+ e} = 2 - \alpha \quad , \quad x^T B(I/J) y = \frac{1 - e^T w}{e^T B(I/J)^+ e} = 1$$

to players I and II respectively.

For $\alpha = 0$ or 1 the set J is $\{2\}$ or $\{1\}$ respectively. This illustrates the nonuniqueness of the equilibrium points and the nonuniqueness of the corresponding payoffs to each player.

3. Any equilibrium point (x, y) corresponds, via the conditions (4)-(11), to two index sets $I \subset M$ and $J \subset N$. The corresponding submatrices $A(I/J)$ and $B(I/J)$, which are the essential parts of the payoff matrices A and B at the given equilibrium point, satisfy (4) and (5) (among other conditions). This fact may be used in excluding certain submatrices from consideration as possible optimal $A(I/J)$ or $B(I/J)$. For instance, in the above example, mixed strategies are excluded for player I since the matrix.

$$A = \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

does not satisfy (4).

4. Extreme equilibrium points, e.g. [6], may be characterized in the same way that basic optimal strategies were characterized in [9] and in corollaries 1 and 2 of [1].

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