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Concerning the Refined Chern Classes of a Holomorphic Vector Bundle

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Geometria differenziale. — *Concerning the Refined Chern Classes of a Holomorphic Vector Bundle.* Nota di BRUNO BIGOLIN (**) e RONNY O. WELLS, Jr., presentata (*) dal Corrisp. A. ANDREOTTI.

RIASSUNTO. — Utilizzando un teorema di immersione dei gruppi di Aeppli nella coomologia a valori complessi, si dimostra che, sotto opportune ipotesi per la base, l'annullamento della classe di Chern di un fibrato olomorfo equivale all'annullarsi della classe raffinata.

Let $E \to X$ be a holomorphic vector bundle and let $c(E)$ be the total Chern class of E in de Rham group $H^*(X, C)$. It is known that $c(E)$ can be defined as follows. Let h be any hermitian metric on E, and let K be the associated curvature tensor, then one defines (with respect to a local frame)

$$c(E, h) = \det(I + (i/2\pi) \cdot K)$$

which defines a d–closed differential form on X composed of homogeneous terms of type (p, p). The image in $H^*(X, C)$ of $c(E, h)$ is shown to be independent of h and this is the total Chern class of E (see e.g. [2]).

Let $A^{p,q}(X)$ be the differential forms on X of type (p, q), $A^r(X) = \sum_{p+q=r} A^{p,q}(X)$, the forms of total degree r, and $A(X) = \Sigma A^r(X)$. Moreover let d be exterior differentiation and let $d = d' + d''$ where

$$d' : A^{p,q}(X) \longrightarrow A^{p+1,q}(X)$$

$$d'' : A^{p,q}(X) \longrightarrow A^{p,q+1}(X)$$

are the usual operators. In [2] it is shown that if we define

$$\Lambda^{p,q}(X) = \frac{\operatorname{Ker} d : A^{p,q}(X) \to A^{p+q+1}(X)}{\operatorname{Im} d'\, d'' : A^{p-1,q-1}(X) \to A^{p,q}(X)} ,$$

then the d–closed form $c(E, h)$ defines an element in the vector space

$$\Lambda(X) = \sum_{q=1}^{n} \Lambda^{q,q}(X),$$

(*) Nella seduta del 19 aprile 1969.
(**) Il primo autore è stato sostenuto dal gruppo di ricerca per la Matematica (ex. n. 35) del C.N.R.

which is independent of the metric h. Let us denote the image by $\hat{c}$ (E). This is the *refined Chern class* of Bott and Chern, and in their paper they prove that $\hat{c}(E \oplus F) = \hat{c}(E) \cdot \hat{c}(F)$, so these are obstructions to splitting off trivial holomorphic subbundles from a given bundle. The question is: how do these obstructions relate to the classical obstructions c (E)? We note that it is trivial that $\hat{c}(E) = 1$ implies that $c(E) = 1$, but the converse is not clear at all. What we shall do is to show that under certain conditions on X the refined Chern classes are precisely the same obstructions as the classical ones, i.e., under certain conditions, the refined Chern classes are *not* refined.

We have two results. The first asserts that for a compact Kähler manifold X, the two obstruction theories agree. In addition we show that for a n–dimensional complex manifold X, we have $c_n(E) = 0$ implies $\hat{c}_n(E) = 0$. Note that $\Lambda(X)$ depends apriori on the analytic structure of X. We shall see that in fact, for certain cases, $\Lambda(X)$ depends only on the topological structure of X. Note that there is a natural map,

$$\gamma_{p,q} : \Lambda^{p,q}(X) \longrightarrow H^{p+q}(X, C).$$

THEOREM 1. *Let* X *be a compact Kähler manifold, then the natural mapping (for all* p, q*)*

$$\gamma_{p,q} : \Lambda^{p,q}(X) \longrightarrow H^{p+q}(X, C),$$

is an injection.

Moreover, $c_j(E) = 0$ *if and only if* $\hat{c}_j(E) = 0$.

THEOREM 2. *Let* X *be a complex manifold with* $\dim_C X = n$. *Then*

$$\gamma_{n,n} : \Lambda^{n,n}(X) \longrightarrow H^{2n}(X, C) \tag{1}$$

is an isomorphism. Consequently, $c_n(E) = 0$ *if and only if* $\hat{c}_n(E) = 0$.

To prove Theorem 1 we first have some preliminary propositions. Using the notation of [3], we define the differential operator (see [1]):

$$D = d'' d' \delta' \delta'' + \delta'' \delta' d' d'' + \delta' d'' \delta'' d' + \delta'' d' \delta' d'' + \delta' d' + \delta'' d'',$$

where δ', δ'' are the adjoints of d', d'' with respect to an Hermitian metric on X. The operator D is homogeneous of bidegree (0, 0) and we have the following norm (Hodge inner product on A (X)),

$$\langle D\varphi, \varphi \rangle = \|\delta' \delta'' \varphi\|^2 + \|d' d'' \varphi\|^2 + \|\delta'' d' \varphi\|^2$$
$$+ \|\delta' d'' \varphi\|^2 + \|d' \varphi\|^2 + \|d'' \varphi\|^2$$

for $\varphi \in A(X)$. Letting

$$H_D^{p,q}(X) = \{\varphi \in A^{p,q}(X) : D\varphi = 0\}$$

be the D–harmonic forms, we have the following

Proposition 1. For any compact complex manifold X and any hermitian metric on X we have

$$\Lambda^{p,q}(X) \cong H_D^{p,q}(X).$$

Proof. One shows easily that D is a selfadjoint elliptic operator, and that there is a Hodge decomposition for the operator, analogous to the classical Laplacian operator, from which is easily derived

$$d' d'' A^{p-1,q-1}(X) \oplus H_D^{p,q} = A^{p,q}(X) \cap \operatorname{Ker} d,$$

where $\oplus$ is the orthogonal direct sum.

Proposition 2. If X is a compact Kähler manifold, then $D = \frac{1}{4}\Delta^2 + \delta' d' + \delta'' d''$, D being computed with respect to the Kähler metric.

Proof. Letting Δ', Δ'' be the d'—and d''—Laplacians respectively, we note that

$$D = \Delta' \Delta'' + \delta' d' + \delta'' d'',$$

using the commutation relations (valid on a Kähler manifold)

$$d' \delta'' = -\delta'' d' \quad , \quad d'' \delta' = -\delta' d'',$$

and the universally valid relations

$$d' d'' + d'' d' = \delta' \delta'' + \delta'' \delta' = 0.$$

But $\Delta = \frac{1}{2}\Delta' = \frac{1}{2}\Delta''$ on a Kähler manifold and Proposition 2 follows.

We are now in a position to prove Theorem 1:

Proof of Theorem 1.

Let D be the operator defined above with respect to the Kähler metric on X, then we have

$$\langle D\varphi , \varphi \rangle = \frac{1}{4}\|\Delta\varphi\|^2 + \|d' \varphi\|^2 + \|d'' \varphi\|^2,$$

so that $D\varphi = 0$ implies $\Delta\varphi = 0$. Conversely, since X is compact Kähler, $\Delta\varphi = 0$ implies that $d' \varphi = d'' \varphi = \delta' \varphi = \delta'' \varphi = 0$ and so $D\varphi = 0$. Thus letting $H_\Delta^{p,q}(X)$ be the Δ–harmonic (p , q) forms on X, we have

$$H_\Delta^{p,q}(X) = H_D^{p,q}(X).$$

But the standard Hodge Theory tells us that

$$H^r(X , C) \cong \sum_{p+q=r} H_\Delta^{p,q}(X),$$

in particular,

$$H_{\Delta}^{p,q}(X) \to H^{p+q}(X, C)$$

is an injection, so

$$\Lambda^{p,q}(X) = H_D^{p,q}(X) \to H^{p+q}(X, C)$$

is an injection. q.e.d.

Proof of Theorem 2.

Consider the exact sequence

$$(2) \qquad O \to C \overset{\alpha}{\Rightarrow} \mathcal{O} \oplus \overline{\mathcal{O}} \overset{\varrho}{\to} \mathcal{H} \to O$$

defined by $\alpha(c) = (c, -c)$, $\beta(f, g) = f + g$ where $\mathcal{H}$ is the sheaf of germs of (complex valued) pluriharmonic functions, $\mathcal{O}$ is the structure sheaf of X, $\overline{\mathcal{O}}$ is the conjugate of $\mathcal{O}$, and C is the constant sheaf of complex numbers on X. We see that (2) is a representation of the assertion that locally any pluriharmonic function is the sum of a holomorphic and antiholomorphic function, and any function which is both holomorphic and antiholomorphic is locally constant.

We have two cases to consider. First for $n = 1$, we note that (1) can be proved by elementary elliptic operator theory but we proceed as follows. First, we have the fine resolution

$$(3) \qquad O \longrightarrow \mathcal{H} \longrightarrow \mathcal{E}^{0,0} \xrightarrow{d'd''} \mathcal{E}^{1,1} \to 0$$

and consequently,

$$\Lambda^{1,1}(X) \cong H^1(X, \mathcal{H}).$$

Moreover, from (2) we have the exact sequence

$$H^1(X, C) \to H^1(X, \mathcal{O}) \oplus H^1(X, \overline{\mathcal{O}}) \to H^1(X, \mathcal{H}) \to H^2(X, C) \to 0.$$

If X is compact, then we have

$$H^1(X, C) \cong H^1(X, \mathcal{O}) \oplus H^1(X, \overline{\mathcal{O}})$$

by standard Riemann surface theory, which implies

$$\Lambda^{1,1}(X) \cong H^1(X, \mathcal{H}) \cong H^2(X, C).$$

and we have (1). If X is open, then $H^1(X, \mathcal{O}) = H^1(X, \overline{\mathcal{O}}) = 0$ (the Mittag-Leffler problem or Cousin I is solvable) and again we have (1).

For $n > 1$, we find in [1], the following resolution of $\mathcal{H}$ (a generalization of (3)), valid on any complex manifold X of complex dimension n. Namely,

$$(4) \qquad 0 \to \mathcal{H} \xrightarrow{j} \mathcal{A}^0 \xrightarrow{h^0} \mathcal{A}^1 \xrightarrow{h^1} ---- \to \mathcal{A}^{2n-2} \xrightarrow{h^{2n-2}} \mathcal{A}^{2n-1} \to 0,$$

where

a) $\mathcal{A}^{2n-2} = \mathcal{E}^{n-1,n-1}$, $\mathcal{A}^{2n-1} = \mathcal{E}^{n,n}$, $h^{2n-2} = d' d''$.

b) $\mathcal{A}^i$ are fine for $i \geq n - 1$.

c) for $i < n - 1$, $\mathcal{A}^i = \mathcal{F} \oplus \mathcal{G} \oplus \mathcal{K}$, where $\mathcal{F}$ is a locally free $\mathcal{O}$–module, $\mathcal{G}$ is a locally free $\bar{\mathcal{O}}$–module, and $\mathcal{K}$ is fine.

There is a spectral sequence

$$H^q(X, \mathcal{O}^p) \Rightarrow H^r(X, \mathcal{H})$$

given by the resolution (4), where

$$E_1^{p,q} = H^q(X, \mathcal{O}^p)$$

and

$$\sum_{p+q=r} E_\infty^{p,q} \simeq H^r(X, \mathcal{H}).$$

We see that $E_2^{2n-1,0} = \Lambda^{n,n}(X)$, since

$$\begin{array}{ccccc} E_1^{2n-2,0} & \xrightarrow{d_1} & E_1^{2n-1,0} & \longrightarrow & 0 \\ \| & & \| & & \\ H^0(X, \mathcal{A}^{2n-2}) & \xrightarrow{d'd''} & H^0(X, \mathcal{A}^{2n-1}) & \longrightarrow & 0 \end{array}$$

and $E_2^{2n-1,0} = E_1^{2n-1,0}/d_1(E_1^{2n-2,0})$.

It follows from *b)* that d_s is trivial, $s \geq n$, and thus $E_\infty^{p,q} = E_n^{p,q}$. Also, from *c)* it follows that

$$E_1^{p,q} = H^q(X, \mathcal{A}^p) = 0, \qquad \text{for} \quad q > n,$$

by a standard theorem in several complex variables. Thus

$$E_n^{p,q} = 0, \qquad (q > n) \quad \text{or} \quad (p \geq n-1, q > 0)$$

$$E_n^{p,q} = 0, \qquad p + q = 2n - 1, q > 0.$$

$$H^{2n-1}(X, \mathcal{H}) \cong \sum_{p+q=2n-1} E_\infty^{p,q} = E_n^{2n-1,0}.$$

$$\cong \Lambda^{n,n}(X).$$

But we have from (2) the exact sequence

$$H^{2n-1}(X, \mathcal{O}) \oplus H^{2n-1}(X, \bar{\mathcal{O}}) \to H^{2n-1}(X, \mathcal{H})$$

$$\to H^{2n}(X, C) \to H^{2n}(X, \mathcal{O}) \oplus H^{2n}(X, \bar{\mathcal{O}}),$$

and

$$H^q(X, \mathcal{O}) = H^q(X, \bar{\mathcal{O}}) = 0, \qquad \text{for} \quad q > n,$$

so

$$H^{2n-1}(X, \mathcal{H}) \cong H^{2n}(X, C)$$

q.e.d.

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