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# On quasi-normalizable operators I

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**Analisi matematica.** — *On quasi-normalizable operators I.* Nota di VASILE ISTRĂŢESCU e IOANA ISTRĂŢESCU, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questa nota si introduce una classe di operatori che generalizzano la classe degli operatori simmetrizzabili e si danno alcune proprietà riguardanti il limite, il raggio spettrale e lo spettro.

1. The aim of this Note is the study of bounded operators over a complex Banach space B which satisfy an additional condition with respect to a scalar product on B. We suppose that there exists a scalar product $\langle\ \rangle$ and a norm $|\ |$ with the property

$$\| x \| \leq k \, | x |$$

where $\| x \| = \langle x , x \rangle^{1/2}$ and $|\ |$ is the initial norm on B.

*Definition* 1.1. An operator $T \in \mathfrak{L}(B)$ is quasi-normalizable if the inequality

$$\| Tx \|^2 \leq \| T^2 x \|$$

is true for every $x$ with $\| x \| = 1$.

In the next section we obtain some properties of this class of operators which are inspired by theorems proved by P. Lax [3]. We remark that the definition of our class is free from the existence of an adjoint of T with respect to the scalar product. The class of operators with property given in definition 1.1. has its origin in the study of some proper class of operators on Hilbert space generalizing hyponormal operators [4].

2. Let B be a Banach space as in § 1.

THEOREM 2.1. *If* T *is quasi-normalizable then* T *is bounded with respect to the Hilbert norm.*

*Proof.* We prove that for every integer $m$

$$\| Tx \|^m \leq \| T^m x \|$$

if $\| x \| = 1$.

For $m = 1$ this is trivial, for $m = 2$ is exactly the definition 1.1. Suppose now, that the inequality is valid for $n = 1 , 2 , \cdots , p$ we show that it is true for $p + 1$. Indeed

$$\| T^{p+1} x \| = \left\| T^p \frac{Tx}{\| Tx \|} \right\| \| Tx \| \geq \| T^2 x \|^p \frac{1}{\| Tx \|^{p-1}} \geq \| Tx \|^{p+1}$$

(*) Nella seduta del 19 aprile 1969.

if $Tx \neq 0$. But, $Tx = 0$ the inequality is clear and thus for every $m$ the inequality is true. Since

$$\| Tx \|^m \leq \| T^m x \| \leq k \,| T^m x | \leq k \,| T^m |\, | x |$$

we obtain that

$$\| Tx \| < | T |$$

and thus

$$\sup_{\|x\|=1} \| Tx \| < | T |$$

and the theorem is proved.

COROLLARY. *The spectral radius* $\nu_T = \lim | T^n |^{1/n} \neq 0$.

*Proof.* We have the inequality (as above)

$$\| Tx \|^m \leq k \,| T^m |\, | x |$$

or

$$\| Tx \| \leq \nu_T$$

For the following theorem, we suppose that T has the property: for every complex number $z$, $Tz = T - zI$ is a quasi-normalizable operator. In this case we can prove.

THEOREM 2.2. *The spectrum of* T *considered on the completion of* B *under norm* $\| \ \|$ *is a subset of* $\sigma(T)$, *the spectrum of* $T \in \mathfrak{L}(B)$.

The proof of this theorem depends on the following lemma which is also of independent interest.

LEMMA. *If* T *is a quasi-normalizable operator and if* $T^{-1} \in \mathfrak{L}(B)$ *the* $T^{-1}$ *is quasi-normalizable.*

*Proof.* of the lemma. Consider $x \in B$, $\| x \| = 1$. We find an element $y \in B$ such that $x = T^2 y$. Since T is quasi-normalizable

$$\| T^{-1} x \|^2 = \| Ty \|^2 = \left\| T \frac{y}{\|y\|} \right\|^2 \| y \|^2 \leq \left\| T^2 \frac{y}{\|y\|} \right\| \| y \|^2 = \| x \| \, \| y \| = \| y \| = \| T^{-2} x \|$$

and the lemma is proved. The idea of the proof is in [5].

*Proof.* of the theorem 2.2. Consider $\lambda$ which is not in $\sigma(T)$ and thus $T - \lambda I$ is quasi-normalizable and invertible operator.

By the lemma $(T - \lambda I)^{-1}$ is also quasi-normalizable and by theorem 2.1. is bounded in the Hilbert norm. It is clear that $(T - \lambda I)\, S$ $(S = (T - \lambda I)^{-1})$ is a bounded operator and on a dense set is equal to identity. Then it is identity on the completion and the theorem is proved.

Suppose now T has the property, there exists an operator $T^* \in \mathfrak{L}(B)$, such that

$$T^* T - T T^* \geq 0. \qquad (*)$$

*Remark.* It is easy to see that every operator with this property is quasi-normalizable.

THEOREM 2.3. *If* T *is compact then* T *is with property*

$$T^* T = TT^*.$$

*Proof.* From the continuity of scalar product it follows that on the completion under the norm $\| \; \|$ we have a compact hyponormal operator. By Ando's theorem [1], [2] we have in fact a normal operator.

*Remark.* The conclusion on the theorem remains valid under weaker condition: the existence of adjoint and that T is quasi-normalizable.

LEMMA. *If* $Tx = \lambda x$ *then* $T^* = \bar{\lambda} x$.

As in [3] for operators with the above property we can prove.

THEOREM 2.4. *The null space of* T *over* B *and over completion under the norm* $\| \; \|$ *is the same.*

In a paper which will follow we study in connections with above consideration: 1) an application to a generalization of Hilbert quadratic form, 2) the essential spectrum for some subclasses of quasi-normalizable operators, namely operators with property.

## REFERENCES.


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