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Tangent flag bundles and generalized Jacobian varieties. Nota I

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Tangent flag bundles and generalized Jacobian varieties.* Nota I di AUBREY WILLIAM INGLETON, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Esposizione riassuntiva di proprietà relative a certe sottovarietà di una varietà algebrica V, introdotte quali varietà jacobiane generalizzate inerenti a nidi di sistemi lineari di ipersuperficie di V, in collegamento alla coomologia delle varietà di bandiere (o nidi di faccette tangenti) della V.

INTRODUCTION.

A comprehensive definition of "Jacobian" subvarieties of an algebraic variety V will be given, involving a number of nests of linear systems of primals on V and contact conditions expressed in terms of tangent flags to V. The definition includes the classical Jacobian in its most general form ([1] p. 22, [2] (1)) and the "generalized Jacobian" of [3] as very special cases. It will be shown the cohomology class of such a "Jacobian" can be computed using the structure of the cohomology ring of the tangent flag bundle V^{Δ} of V, and an explicit formula for the cohomology class will be obtained in a comparatively simple case which is still very much wider than the classical.

The present Note I is devoted to a description of the cohomology rings of flag manifolds and tangent flag bundles and in particular to the geometrical interpretation of the generators of those rings which appear in the

(*) Nella seduta dell'8 marzo 1969.

(1) References are given at the end of Note II.

Borel-Hirzebruch theory ([4], [5], [6]). The following Note II is concerned with subvarieties of V^{Δ} associated with nests of linear systems on V and their use in computing Jacobians.

These Notes only contain a summary of the ideas and principal results involved and all proofs are omitted. A full account will be published later.

NOTATION. For any elements $x_0, \cdots, x_n$ of a ring with unity, $\sigma_h(x_0, \cdots, x_n)$ will denote the value of the elementary symmetric polynomial of degree h in $x_0, \cdots, x_n$ and is to be interpreted as 1 when $h = 0$ and as zero when $h < 0$ or $h > n + 1$; $\bar{\sigma}_h(x_0, \cdots, x_n)$ will denote the value of the complete symmetric polynomial of degree h in $x_0, \cdots, x_n$ and is to be interpreted as 1 when $h = 0$ and as zero when $h < 0$.

FLAG MANIFOLDS AND TANGENT FLAGS.

1.0. A (complex) flag is a nest of projective subspaces

$$S : S_0 \subset S_1 \subset \cdots \subset S_{n-1} \quad , \quad \dim S_i = i,$$

of complex projective space $P_n(\mathbf{C})$. The set $F = F(n+1)$ of all such flags (in a given projective space) may be regarded in a natural way as an algebraic variety. It may also be identified with the space of cosets of the full linear group $GL(n+1, \mathbf{C})$ modulo the subgroup $\Delta(n+1, \mathbf{C})$ of triangular matrices. This structure defines a $\Delta(n+1, \mathbf{C})$–bundle ξ over F.

For $j = 0, \cdots, n$, the natural homomorphism

$$p_j : \Delta(n+1, \mathbf{C}) \to GL(j+1, \mathbf{C})$$

(restriction to the first $j+1$ rows and columns) determines a $(j+1)$–dimensional vector bundle $p_j\xi$ over F. Let ξ_j be the quotient line-bundle $p_j\xi/p_{j-1}\xi$ $(j = 1, \cdots, n)$ and put $\xi_0 = p_0\xi$. Then the cohomology ring $H^*(F) = H^*(F, \mathbf{Z})$ is generated by the first Chern classes $\gamma_j = c_1(\xi_j) \in H^2(F)$ subject to the relations

$$\sigma_h(\gamma_0, \cdots, \gamma_n) = 0 \qquad (h = 1, \cdots, n+1),$$

i.e.

$$\prod_{j=0}^{n}(1+\gamma_j) = 1 \tag{1.0.1}$$

([6] p. 106).

1.1. INCOMPLETE FLAGS. If $q_0, \cdots, q_m$ are integers,

$$0 \le q_0 < q_1 < \cdots < q_m = n,$$

a $(q_0, \cdots, q_m)$–flag is a nest of projective subspaces

$$S : S_{q_0} \subset S_{q_1} \subset \cdots \subset S_{q_{m-1}} \quad , \quad \dim S_{q_i} = q_i,$$

of $P_n(\mathbf{C})$. The set $W = W(q_0, \cdots, q_m)$ of all such flags (for given $q_0, \cdots, q_m$) is an algebraic variety; the integers $q_0, \cdots, q_m$ will be called the *flag-dimensions* of W.

There is a natural projection

$$\pi : \mathrm{F} \to \mathrm{W}, \text{ fibre } \mathrm{F}(q_0+1) \times \mathrm{F}(q_1 - q_0) \times \cdots \times \mathrm{F}(q_m - q_{m-1}),$$

which induces a monomorphism

$$\pi^* : \mathrm{H}^*(\mathrm{W}) \to \mathrm{H}^*(\mathrm{F}).$$

The image $\pi^* \mathrm{H}^*(\mathrm{W})$ consists of those elements of $\mathrm{H}^*(\mathrm{F})$ which are symmetrical in all the pairs (γ_j, γ_{j+1}) with j *not* a flag-dimension of W ([4] p. 202).

If j and (unless $j = 0$) $j - 1$ are flag-dimensions of W then $\gamma_j \in \pi^* \mathrm{H}^2(\mathrm{W})$ and its inverse image in $\mathrm{H}^2(\mathrm{W})$ will be denoted by $\gamma_j(\mathrm{W})$. If p, q are flag-dimensions of W, $p < q$, the inverse images in $\mathrm{H}^{2h}(\mathrm{W})$ of $\sigma_h(\gamma_{p+1}, \cdots, \gamma_q)$, $(-1)^h \bar{\sigma}_h(\gamma_{p+1}, \cdots, \gamma_q)$ will be denoted by $\sigma_h(p, q; \mathrm{W})$, $\sigma_h(q, p; \mathrm{W})$ respectively. In view of (1.0.1) we have

$$\pi^* \sigma_h(q, p; \mathrm{W}) = \sigma_h(\gamma_{q+1}, \cdots, \gamma_n, \gamma_0, \cdots, \gamma_p).$$

(1.0.1) also implies that the relations

$$\sigma_h(p, q; \mathrm{W}) = \sum_{i=0}^{h} \sigma_i(p, r; \mathrm{W})\, \sigma_{h-i}(r, q; \mathrm{W}) \tag{1.1.1}$$

hold for *any* three flag-dimensions p, q, r (irrespective of order).

1.2. THE EHRESMANN BASE. The term *index*, or more precisely (h, n)–index, will mean an $(h+1)$–tuple

$$\boldsymbol{k} = (k_0, \cdots, k_h)$$

of distinct integers, $0 \leq k_i \leq n$ $(i = 0, \cdots, h)$. For each $j = 1, \cdots, n$, let $\mathrm{Q}_j(\boldsymbol{k})$ be the set of integers q such that

(i) $j \in \{k_0, k_1, \cdots, k_q\}$,
(ii) $j - 1 \notin \{k_0, k_1, \cdots, k_q\}$,
(iii) $k_q \geq j$,
(iv) $k_{q+1} < j$ (or $q = h$).

Then, for each $q \in \mathrm{Q}_j(\boldsymbol{k})$, let $d_j(q; \boldsymbol{k})$ be the number of integers in $\{0, \cdots, j\}$ which are not in $\{k_0, \cdots, k_q\}$.

Relative to a fixed flag

$$\mathrm{E} : \mathrm{E}_0 \subset \mathrm{E}_1 \subset \cdots \subset \mathrm{E}_{n-1}$$

in $\mathrm{P}_n(\mathbf{C})$ the *Ehresmann subvariety* $[\boldsymbol{k}; \mathrm{F}]$ of $\mathrm{F} = \mathrm{F}(n+1)$ is defined for any (h, n)–index $\boldsymbol{k}$, $0 \leq h \leq n$, as consisting of all the flags satisfying the conditions

$$\dim(\mathrm{S}_q \cap \mathrm{E}_{n-j}) \geq d_j(q; \boldsymbol{k}) + q - j \qquad (j = 1, \cdots, n;\ q \in \mathrm{Q}_j(\boldsymbol{k})). \tag{1.2.1}$$

The cohomology class dual to $[\boldsymbol{k}; F]$ will be denoted by $[\boldsymbol{k}; F]^*$ and is independent of E. We observe that conditions are imposed only on the flag-components with dimensions belonging to the set

$$Q(\boldsymbol{k}) = \bigcup_{j=1}^{n} Q_j(\boldsymbol{k}).$$

The classes $[\boldsymbol{k}; F]^*$ form a base for cohomology of F (cfr. [7], [8]–but the integers in an index represent *co*dimensions whereas Ehresmann's and Monk's symbols use actual dimensions; this modification is essential for the development of an analogous notation in connection with nests of linear systems in 2.0 infra). The correspondence between indices and Ehresmann varieties is not one-one: clearly, when $h > 0$, $[\boldsymbol{k}; F] = [k_0, \cdots, k_{h-1}; F]$ if (and only if) $h \notin Q(\boldsymbol{k})$. Each Ehresmann variety can be represented by a unique (n, n)–index (permutation—as in [8]); alternatively (and more appropriately in § 2) by a unique *proper* index, i.e. $h \in Q(\boldsymbol{k})$ (or $\boldsymbol{k} = (0)$).

For an incomplete-flag manifold $W = W(q_0, \cdots, q_m)$, it is clear that $[k; F]^* \in \pi^* H^*(W)$ if and only if $Q(\boldsymbol{k}) \subseteq \{q_0, \cdots, q_{m-1}\}$. The inverse image in $H^*(W)$ will then be denoted by $[\boldsymbol{k}; W]^*$; the set of all such cohomology classes forms a base fo $H^*(W)$ [7].

If q is a flag-dimension of W and in particular if $W = F$, we write, for $r = 0, \cdots, q+1$ and $s = 0, \cdots, n-q$,

$$(1.2.2) \qquad \omega_{r,s}(q; W) = [0, 1, \cdots, q-r, q-r+s+1, \cdots, q+s; W]^*$$

This is an element of $H^{2rs}(W)$; it is the unit element if r or s is zero, otherwise it corresponds to the single condition

$$\dim(S_q \cap E_{n-q+r-s-1}) \geq r - 1.$$

It will also be necessary sometimes to regard $\omega_{r,s}(q; W)$ as defined and equal to zero for values of r or s outside the ranges stated. Finally we write

$$(1.2.3) \qquad \omega(q; W) = \omega_{1,1}(q; W).$$

1.3. DUALITY. Let

$$\tau : GL(n+1, \mathbf{C}) \to GL(n+1, \mathbf{C})$$

be the automorphism defined by

$$\tau A = J A^{-1\mathsf{T}} J,$$

where J is the $(n+1)\times(n+1)$ matrix

$$\begin{bmatrix} 0 & & \cdot^{\,1} \\ & \cdot & \\ \cdot_{\,1} & & 0 \end{bmatrix},$$

and T denotes transposition.

The subgroup $\Delta(n+1, \mathbf{C})$ is stable under τ and so there is an induced map

$$\tau_F : F(n+1) \rightarrow F(n+1),$$

and $\tau_F^2 = 1$. We call τ_F the *duality map*. If

$$W = W(q_0, \cdots, q_{m-1}, n) \quad \text{and} \quad \overline{W} = W(n - q_{m-1} - 1, \cdots, n - q_0 - 1, n),$$

there is also an induced duality map

$$\tau_W : W \rightarrow \overline{W}$$

such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\tau_F} & F \\ \pi \downarrow & & \downarrow \overline{\pi} \\ W & \xrightarrow{\tau_W} & \overline{W} \end{array}$$

commutes.

It is readily verified that

$$\tau_F^* \xi_j = \xi_{n-j}^{-1},$$

so that

$$\tau_F^* \gamma_j = -\gamma_{n-j}$$

and

$$\sigma_h(p, q; W) = (-1)^h \tau_W^* \sigma_h(n-q-1, n-p-1; \overline{W})$$

for any two flag-dimensions p, q of W.

The effect on Ehresmann classes is

$$[k_0, \cdots, k_h; W]^* = \tau_W^* [n - k_h, \cdots, n - k_0; \overline{W}]^*$$

whenever $Q(\boldsymbol{k}) \subseteq \{q_0, \cdots, q_{m-1}\}$. In particular, if q is a flag-dimension of W,

$$\omega_{r,s}(q; W) = \tau_W^* \omega_{s,r}(n-q-1; \overline{W}).$$

1.4. Using Monk's intersection formula ([8] Theorem 3 and cfr. 2.4 infra) for

$$\omega(q; F)[\boldsymbol{k}; F]^*$$

and the relations

$$\omega(q; F) = -(\gamma_0 + \cdots + \gamma_q)$$

([8] (13.5), [3] (2.4.2)) it is possible in principle to express any polynomial in the γ_j as a linear combination of Ehresmann classes and conversely to express any Ehresmann class as a polynomial in the γ_j. No explicit general formulae appear to be known. However, it is possible to prove indirectly that

$$\sigma_h(q, n; \mathrm{F}) = \omega_{1,h}(q; \mathrm{F}) \tag{1.4.1}$$

and dually

$$\sigma_h(n, q; \mathrm{F}) = (-1)^h \omega_{h,1}(q; \mathrm{F}), \tag{1.4.1'}$$

from which it follows using (1.1.1) that

$$\sigma_h(p, q; \mathrm{F}) = \sum_{r=0}^{h} (-1)^r \omega_{r,1}(q; \mathrm{F}) \, \omega_{1,h-r}(p; \mathrm{F}).$$

(Each product on the right is in fact an Ehresmann class if $0 \leq h \leq q - p$ or if $p > q + 1$).

If $\boldsymbol{k} = (k_0, \cdots, k_q)$ where $k_0 < k_1 < \cdots < k_q$, then $\mathrm{Q}(\boldsymbol{k}) = \{q\}$ and so $[\boldsymbol{k}; \mathrm{F}]^* \in \pi^* \mathrm{H}^*(\Omega)$ where Ω is the Grassmannian $\mathrm{W}(q, n)$. We can then express $[\boldsymbol{k}; \mathrm{F}]^*$ as a polynomial in $\gamma_0, \cdots, \gamma_q$ using the formula

$$[\boldsymbol{k}; \Omega]^* = \begin{vmatrix} \overline{\omega}_{k_0} & \overline{\omega}_{k_0-1} \cdots \overline{\omega}_{k_0-q} \\ \overline{\omega}_{k_1} & \overline{\omega}_{k_1-1} \cdots \overline{\omega}_{k_1-q} \\ \cdots & \cdots \\ \overline{\omega}_{k_q} & \overline{\omega}_{k_q-1} \cdots \overline{\omega}_{k_q-q} \end{vmatrix}, \tag{1.4.2}$$

where $\overline{\omega}_h$ denotes $\omega_{1,h}(q; \Omega)$, ([9] p. 358, with k_i replacing $n - a_{n-1}$). Hence, using (1.4.1),

$$[\boldsymbol{k}; \mathrm{F}]^* = \begin{vmatrix} \tilde{\sigma}_{k_0} \cdots \tilde{\sigma}_{k_0-q} \\ \cdots \\ \tilde{\sigma}_{k_q} \cdots \tilde{\sigma}_{k_q-q} \end{vmatrix}, \tag{1.4.3}$$

where $\tilde{\sigma}_h$ denotes $\sigma_h(\gamma_{q+1}, \cdots, \gamma_n) = (-1)^h \overline{\sigma}_h(\gamma_0, \cdots, \gamma_q)$. In particular

$$\omega_{r,s}(q; \mathrm{F}) = \begin{vmatrix} \tilde{\sigma}_s & \tilde{\sigma}_{s-1} \cdots \tilde{\sigma}_{s-r+1} \\ \tilde{\sigma}_{s+1} & \tilde{\sigma}_s \cdots \tilde{\sigma}_{s-r+2} \\ \cdots & \cdots \\ \tilde{\sigma}_{s+r-1} & \tilde{\sigma}_{s+r-2} \cdots \tilde{\sigma}_s \end{vmatrix},$$

or dually

$$(-1)^{rs} \omega_{r,s}(q; \mathrm{F}) = \begin{vmatrix} \sigma_r \cdots \sigma_{r-s+1} \\ \cdots \\ \sigma_{r+s-1} \cdots \sigma_r \end{vmatrix},$$

where σ_h denotes $\sigma_h(\gamma_0, \cdots, \gamma_q)$.

We shall be especially interested in the case $r = q$; the last determinant can then be simplified to give

$$(-1)^{qs} \omega_{q,s}(q; \mathrm{F}) = (\gamma_0 \gamma_1 \cdots \gamma_q)^s \, \overline{\sigma}_s\left(\frac{1}{\gamma_0}, \cdots, \frac{1}{\gamma_q}\right) = \hat{\sigma}_s(\gamma_0, \cdots, \gamma_q) \text{ (say)}. \tag{1.4.4}$$

1.5. TANGENT FLAG BUNDLES. Let V be a non-singular algebraic variety of dimension d, which to begin with we suppose to be imbedded in $P_n(\mathbf{C})$. A *tangent flag* to V is a $(0, 1, \cdots, d, n)$–flag S with $S_0 \in V$ and S_d the tangent $[d]$ to V at S_0. The set of all such flags is an algebraic variety, the *tangent flag bundle* V^Δ of V. There is a natural injection

$$\theta : V^\Delta \to W = W(0, 1, \cdots, d, n),$$

and a natural projection

$$\rho : V^\Delta \to V, \qquad \text{fibre } F(d)$$

which induces a monomorphism

$$\rho^* : H^*(V) \to H^*(V^\Delta).$$

$H^*(V^\Delta)$ is generated by $\rho^* H^*(V)$ and elements $\delta_1, \cdots, \delta_d \in H^2(V^\Delta)$, subject to the relations

$$\sigma_h(\delta_1, \cdots, \delta_d) = (-1)^h \rho^* c_h(V) \quad (h = 1, \cdots, d)$$

where $c_h(V)$ is the hth Chern class of V; i.e.

$$\prod_{j=1}^{d} (1 - \delta_j) = \rho^*(1 + c_1(V) + \cdots + c_d(V)) = \rho^* c(V) \tag{1.5.1}$$

(cfr. [6] 4.2, 13.3, but we are using generators with the opposite sign to simplify later calculations). For our purposes the δ_j may be identified by

$$\begin{aligned}\delta_j &= \theta^*(\gamma_0(W) - \gamma_j(W)) \\ &= \theta^*(\omega(j; W) - \omega(j-1; W) - \omega(0; W))\end{aligned}$$

(cfr. [6] Satz 13.1.1 and [3] 3.5).

The *incomplete tangent flag bundle* $T = T(q_1, \cdots, q_m; V)$, $1 \leq q_1 < \cdots < q_m = d$, is the variety of all tangent $(0, q_1, \cdots, q_m, n)$–flags to V. The integers $q_1, \cdots, q_m$ will be called the *flag-dimensions* of T. There is a natural injection

$$T \to W(0, q_1, \cdots, q_m, n),$$

a natural projection

$$T \to V, \qquad \text{fibre } W(q_1 - 1, \cdots, q_m - 1),$$

and a natural projection

$$\pi : V^\Delta \to T, \text{ fibre } F(q_1) \times F(q_2 - q_1) \times \cdots \times F(q_m - q_{m-1}),$$

which induces a monomorphism

$$\pi^* : H^*(T) \to H^*(V^\Delta).$$

The image $\pi^* H^*(T)$ consists of those elements of $H^*(V^\Delta)$ which are symmetrical in all the pairs (δ_j, δ_{j+1}) with j not a flag dimension on T.