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On a class of operators

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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi funzionale. — *On a class of operators.* Nota di GHEORGHE CONSTANTIN, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questa nota si considerano degli operatori lineari e limitati per i quali $H^m J - JH^m = iC_m$, $C_m \geq 0$, $m \in N$. Si dimostra che s'esiste un operatore lineare S con $0 \notin cl(W(S))$ e se 1) $ST^p = T^{*p}S$, 2) se $1 + \frac{\lambda}{\bar{\mu}} + \cdots + \left(\frac{\lambda}{\bar{\mu}}\right)^{p-1} \neq 0$ per $\lambda, \mu \in \sigma(T)$, allora T è autoaggiunto.

1. A bounded linear operator T defined on a Hilbert space $\mathcal{H}$ is said to be hyponormal if for every $x \in \mathcal{H}$, $\|T^* x\| \leq \|Tx\|$, or equivalently if $T^* T - TT^* \geq O$. The notion of hyponormality was introduced in [2] through under another name. In [6], [3] was introduced a new class of operators generalizing hyponormal operators.

In this Note we give some properties for a new class of operators which was introduced in [5] by following

DEFINITION.—*An operator* T *is said to be hyponormal of order m if*

$$H^m J - JH^m = iC_m \quad , \quad T = H + iJ$$

where C_m *is a positive operator for some non-negative integer m.*

2. The numerical range of an operator T is the set

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

It is well-known that W(T) is convex and its closure ($cl(W(T))$) contains the spectrum $\sigma(T)$ of T.

(*) Nella seduta dell'8 febbraio 1969.

LEMMA 1.—*If* T *is a hyponormal operator of order* m *such that* $\sigma(T)$ *is a set of real numbers, then* T *is self-adjoint.*

Proof.—It is known that [5, Theorem 5.1]: if $T = H + iJ$ is hyponormal of order m then

$$\sigma(H) \subseteq Pr^x(\sigma(T))$$

$$\sigma(J) \subseteq Pr^y(\sigma(T)).$$

Hence $\sigma(J) = \{0\}$ and it follows that $J = O$.

THEOREM 1.—*Let* T *be a hyponormal operator of order* m. *If for arbitrary operator* S *for which* $0 \notin cl(W(S))$, $ST = T^* S$ *then* T *is self-adjoint.*

Proof.—From Theorem 1 [8] we conclude that the spectrum of T is real and Lemma 1 implies that T is a self-adjoint operator.

We recall that a unitary operator U is cramped if $\sigma(U)$ is contained on an arc of the unit circle with central angle less than π.

COROLLARY.—*If* T *is a hyponormal operator of order* m *which is unitarily equivalent to its adjoint by a cramped unitary operator* U, *then* T *is self-adjoint.*

The proof follows from the fact that $cl(W(U))$ is the convex hull of the spectrum of U and $0 \notin cl(W(U))$.

THEOREM 2.—*If* T *is a hyponormal operator of order* m, S *is an arbitrary operator for which* $0 \notin cl(W(S))$ *and*

1) $ST^p = T^{*p} S$

2) if $\lambda, \mu \in \sigma(T)$, $1 + \frac{\lambda}{\bar{\mu}} + \left(\frac{\lambda}{\bar{\mu}}\right)^2 + \cdots + \left(\frac{\lambda}{\bar{\mu}}\right)^{p-1} \neq 0$ *then* T *is self-adjoint.*

Proof.—Since $T = H + iJ$ is hyponormal of order m it follows that $T_m = H^m - iJ$ is hyponormal. We prove that $\sigma(T_m)$ is real.

Suppose the contrary and let $a + ib = \lambda_0 \in \sigma(T_m)$ with $b \neq 0$. Since $\sigma(J) = Pr^y(\sigma(T_m))$ [6, Theorem 1] we have that $b \in \sigma(J)$ and since $\sigma(J) \subseteq Pr^y(\sigma(T))$, $\sigma(H) \subseteq Pr^x(\sigma(T))$, [5, Theorem 5.1] then there exists $a_1 \in \sigma(H)$ such that

$$(H - a_1 I) x_n \to 0 \quad , \quad (J - bI) x_n \to 0 \quad , \quad \|x_n\| = 1$$

which implies that

$$(T - (a_1 + ib) I) x_n \to 0$$

and

$$(T^* - (a_1 - ib) I) x_n \to 0.$$

Therefore if $\lambda_1 = a_1 + ib$ we have

$$T^p x_n - \lambda_1^p I x_n \to 0$$

$$T^{*p} x_n - \bar{\lambda}_1^p I x_n \to 0$$

and since $0 \notin cl(W(S))$, by 1) we obtain

$$ST^p S^{-1} x_n - \bar{\lambda}_1^p SS^{-1} x_n \to 0$$

or

$$T^p S^{-1} x_n - \bar{\lambda}_1^p S^{-1} x_n \to 0.$$

If we put

$$S^{-1} x_n = y_n$$

it follows

$$(T^{p-1} + \cdots + \bar{\lambda}_1^{p-1})(T - \bar{\lambda}_1) y_n \to 0$$

and from 2) we have

$$(T - \bar{\lambda}_1 I) y_n \to 0.$$

From the identity

$$(\lambda_1 - \bar{\lambda}_1)\langle x_n , y_n\rangle = \langle(\lambda_1 - T) x_n , y_n\rangle + \langle(T - \bar{\lambda}_1) x_n , y_n\rangle =$$
$$= \langle(\lambda_1 - T) x_n , y_n\rangle + \langle x_n , (T^* - \lambda_1) y_n\rangle \to 0$$

we obtain

$$\langle x_n , y_n\rangle \to 0$$

or

$$\left\langle S \frac{S^{-1} x_n}{\|S^{-1} x_n\|} , \frac{S^{-1} x_n}{\|S^{-1} x_n\|} \right\rangle \to 0$$

which represents a contradiction. Hence $\sigma(J) = \{0\}$ and thus T is self-adjoint.

We recall that a Riesz operator is an operator T which has the property that its spectrum consists of an at most denumerable sequence (λ_n) of eigenvalues $\neq 0$, and of zero, which is the limit of (λ_n), when that sequence is infinite; furthermore, for each n, $\mathcal{H}$ is the direct sum of a finite dimensional subspace N_n and a closed subspace F_n, such that both are stable under T, $\lambda_n I - T$ is nilpotent in N_n (which contains the eigenspace E_n corresponding to λ_n), and $\lambda_n I - T$ is an automorphism of F_n.

PROPOSITION 1.—*If* $T = H + iJ$ *is a Riesz operator and hyponormal of order m then*

$$H^m J = JH^m.$$

Proof.—Since the operator $T_m = H^m - iJ$ is hyponormal and [5, Theorem 5.1], $\sigma(J) \subseteq Pr^y(\sigma(T))$, it follows that $\sigma(J)$ has at most one limit point and similarly for $\sigma(H^m)$, we conclude that the operator T_m is hyponormal with a single limit point in its spectrum. Therefore we obtain that T_m is normal and the proposition is proved.

3. In this section we give an application of a result which appears in [8].

PROPOSITION 2.—*Let* A *and* B *a self-adjoint operators; if* A *is positive and has an inverse, then the product* AB *is a spectral operator of scalar type, with real spectrum.*

Proof.—Since A is an invertible and positive operator, (which is the same as the fact that there is an $a > 0$ such that $\langle Tx, x\rangle \geq a\langle x, x\rangle$ for any $x \in \mathcal{H}$) we have that $0 \notin cl(W(A))$.

But

$$AB = ABAA^{-1}$$

and therefore AB is similar to their adjoint. By Theorem 2 [8] AB is similar to a self-adjoint operator. Then AB is scalar with real spectrum.

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