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Characterization of the integral of set-valued functions

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Teoria dell'integrale. — *Characterization of the integral of set-valued functions.* Nota di FRANCESCO S. DE BLASI e ANDRZEJ LASOTA, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostra che esiste una sola possibilità di definire un integrale per funzioni multivoche soddisfacente alcune naturali condizioni (cfr. Th. 1).

The present paper is devoted to the study of the Aumann integral of set-valued functions. We shall show that there is only one possibility of defining an integral for set-valued functions which satisfies some natural conditions (see Theorem 1). By this way we shall obtain a new proof of the statement that the Hukuhara integral, if it exists, coincides with the integral of Aumann.

Let D be a measurable set of R^l with finite Lebesgue measure, $\mu(D) < +\infty$. The family of all non empty subsets of R^m will be denoted by $s(R^m)$. By $c(R^m)$, $(k(R^m))$ will be denoted the metric space of all non empty compact, (compact convex) subsets of R^m with Hausdorff metric τ.

A map $F : D \to s(R^m)$ will be called measurable if it is measurable in the sense of Pliś, i.e., for every $A \in c(R^m)$ the set $\{x \in D : F(x) \cap A \neq \emptyset\}$ is Lebesgue measurable.

By X we denote the space of all functions $F : D \to s(R^m)$; by X_c the subspace of X of all measurable functions $F : D \to c(R^m)$ such that $\int_D |F(x)| \, dx < +\infty$, $(|A| = \tau(A, 0))$. The subspace of X_c which contains all functions $F : D \to k(R^m)$ will be denoted by X_k. The subspace of X_c, (X_k) of all step functions, i.e., of functions F of the form:

$$F = \sum_{i=1}^{n} \chi_{D_i} A_i \ ^{(1)} \quad , \quad \bigcup_{i=1}^{n} D_i = D \, , \, D_i \cap D_j = \emptyset \quad \text{for } i \neq j$$

where χ_{D_i} is the characteristic function of D_i and $A_i \in c(R^m)$, $(k(R^m))$ will be denoted by X_{cs}, (X_{ks}). For $F, G \in X_c$, we set

$$\text{Dist}(F, G) = \int_D \tau(F(x), G(x)) \, dx.$$

PROPOSITION 1.—*The spaces* (X_c, Dist), (X_k, Dist) *are complete metric spaces.*

The fact that Dist is a metric function is quite easy. The proof of the completeness may be carried out by standard arguments (see [2]).

(*) Nella seduta dell'8 febbraio 1969.

(1) $\sum_{i=1}^{n} \lambda_i B_i$ denotes the set $\left\{ \sum_{i=1}^{n} \lambda x_i : x_i \in B_i \right\}$.

PROPOSITION 2.—*The space* X_{cs} *is dense in* X_c. *Moreover, for every function* $F \in X_c$ *there is a sequence of* $F_n \in X_{cs}$ *such that*

$$|F_n(x)| \leq m(x), \; m \text{ integrable} \quad , \quad \lim \tau(F_n(x), F(x)) = 0 \text{ almost everywhere.}$$

Proof.—Let $\bar{X}_{cs}$ be the closure of X_{cs}. Since $\bar{X}_{cs} \subset X_c$ we need only to prove that $X_c \subset \bar{X}_{cs}$. Let us consider $F \in X_c$. By Pliś theorem [5], for each integer $n > 0$ there exists a closed (and bounded) set $D_n \subset D$ such that

$$\mu(D \setminus D_n) < 1/n \tag{1}$$

and such that the restriction F_{D_n} of F to D_n is continuous. We can also suppose that $D_{n+1} \supset D_n$ for each n. For each n there exists a finite partition $\{D_n^i\}$, $i = 1, 2, \cdots, i(n)$ of D_n such that

$$\tau(F(x), F(x_i)) < 1/n \quad , \quad \text{for each } x \in D_n^i, \tag{2}$$

where x_i is a fixed point of D_n^i. The map F_n defined for each x by

$$F_n(x) = \begin{cases} F(x_i) & \text{if } x \in D_n^i, \; i = 1, 2, \cdots, i(n) \\ 0 & \text{if } x \in D \setminus D_n \end{cases}$$

belongs to X_{cs}. Hence, we have $|F_n(x)| \leq |F(x)| + 1$ and

$$\operatorname{Dist}(F_n, F) = \int_{D \setminus D_n} \tau(F(x), 0)\, dx + \sum_{i=1}^{i(n)} \int_{D_n^i} \tau(F(x), F(x_i))\, dx. \tag{3}$$

From (3) by inequalities (1), (2) and the absolute continuity of the integral follows that $\operatorname{Dist}(F_n, F) \to 0$. Thus to complete the proof it is sufficient to put $m = |F| + 1$ and to replace $\{F_n\}$ by a suitably chosen subsequence.

If $F \in X_c$, then the map G from D into $k(R^m)$ given for each x by $G(x) = \overline{co}\, F(x)$, is a measurable function and $|F(x)| = |G(x)|$. So $\overline{co}$ can be considered as a map from X_c into X_k.

PROPOSITION 3.—*The map* $\overline{co}: X_c \to X_k$ *is a continuous surjection. Moreover,*

$$\operatorname{Dist}(\overline{co}\, F, \overline{co}\, G) \leq \operatorname{Dist}(F, G). \tag{4}$$

Proof.—It is easy to see that $\overline{co}$ is onto. Inequality (4) follows from the inequality $\tau(\overline{co}\, A, \overline{co}\, B) \leq \tau(A, B)$ which is valid for all $A, B \in c(R^m)$.

PROPOSITION 4.—X_{ks} *is dense in* X_k. *Moreover, for every* $F \in X_k$, *there is a sequence of* $F_n \in X_{ks}$ *such that*

$$|F_n(x)| \leq m(x), \; m \text{ integrable}, \; \tau(F_n(x), F(x)) \to 0 \text{ almost everywhere.}$$

Proof.—By Proposition 1, $X_k \supset \bar{X}_{ks}$ where $\bar{X}_{ks}$ denotes the closure of X_{ks}. Furthermore Propositions 2 and 3 imply $X_k \subset \bar{X}_{ks}$ and so the first part of the assertion. To prove the second one it is sufficient to observe that the conditions $\tau(F_n(x), F(x)) \to 0$, $F_n \in X_{cs}$, $|F_n| \leq m$ imply $\tau(\overline{co}\, F_n(x), \overline{co}\, F(x)) \to 0$, $\overline{co}\, F_n \in X_{ks}$, $|\overline{co}\, F_n| \leq m$.

Let us recall the definition of the Aumann integral. For $F \in X$ we set

$$J(F) = \left\{ \int_D f(x)\, dx : f(x) \in F(x),\ f \text{ integrable} \right\}.$$

We shall use the following known properties of J, (see [1], [3]).

PROPOSITION 5.—*For every* $F \in X_c$, $J(F)$ *is a closed convex set in* R^m.

PROPOSITION 6.—*The Aumann integral is an additive function, i.e., for* $F \in X$ *and for each finite partition* $\{D_i\}$ *of* D, *we have*

$$J(F) = \sum_{i=1}^{n} J(\chi_{D_i} F).$$

PROPOSITION 7.—J *is a continuous map from* X_k *onto* $k(R^m)$.

PROPOSITION 8.—*For every* $F \in X_c$, *we have*

$$J(F) = J(\overline{co}\, F).$$

From Propositions 8, 7 and 3 we have the following

PROPOSITION 9.—J *is a continuous map from* X_c *onto* $k(R^m)$.

From Propositions 9 and 6 we obtain the following

PROPOSITION 10.—*The Aumann integral is a complete additive function,* i.e., *for every* $F \in X$

$$J(F) = \sum_{i=1}^{\infty} J(\chi_{D_i} F)$$

where $\{D_i\}$ *is a countable partition of* D.

From the definition and Proposition 8 we have the following

PROPOSITION 11.—*For every* F *constant,* $F(x) \equiv A$, $A \in c(R^m)$, *we have*

$$J(A) = \mu(D)\, \overline{co}\, A.$$

We shall show that properties 6, 9 and 11 determine uniquely the integral of set-valued functions. Namely we shall prove the following

THEOREM 1.—*Let* I *be a map from* X_c *into* $c(R^m)$ *such that*

(i) I *is an additive function,* i.e.,

$$I(F) = \sum_{i=1}^{n} I(\chi_{D_i} F) \text{ for every finite partition } \{D_i\} \text{ of } D$$

(ii) I *is a continuous function from* X_c *into* $c(R)^m$

(iii) *For every* $D_0 \subset D$, D_0 *measurable, and every constant* $F(x) \equiv \equiv A$, $A \in c(R^m)$ *the inequality*

$$\mu(D_0)\, A \subset I(\chi_{D_0} A) \subset \mu(D_0)\, \overline{co}\, A \tag{5}$$

holds. Then $I(F) = J(F)$ *for every* $F \in X_c$.

Proof.—We want to prove first that (i) and (ii) imply

$$I(\chi_{D_0} A) = \mu(D_0)\, \overline{co}\, A \tag{6}$$

for each $A \in c(R^m)$ and each measurable $D_0 \subset D$. Since by inequality (5), $I(\chi_{D_0} A) \subset \mu(D_0)\, \overline{co}\, A$, we need only to prove the opposite inequality. To this end let x be a point of $\mu(D_0)\, \overline{co}\, A$. From the compactness of A it follows that x can be expressed by

$$x = \sum_{k=1}^{n} \lambda_k x_k \,,\; x_k \in A \quad , \quad \sum_{k=1}^{n} \lambda_k = 1 \,,\; \lambda_k \geq 0. \tag{7}$$

Let us consider a partition $\{D_k\}$ of D_0 by measurable sets D_k such that

$$\mu(D_k) = \mu(D_0)\, \lambda_k\,. \tag{8}$$

Using (i) and (iii) we obtain $I(\chi_{D_0} A) \supset \sum_{k=1}^{n} \mu(D_k) A$ and, by (8),

$$I(\chi_{D_0} A) \supset \mu(D_0) \sum_{k=1}^{n} \lambda_k A\,.$$

Conditions (7) and the last inequality yield $x \in I(\chi_{D_0} A)$. So the proof of (6) is finished. From (6), Propositions 11, 6 and (i) it follows that the restrictions of I and J to X_{cs} are equal. Proposition 7, (ii) and Proposition 2 imply that I and J coincide on X_c.

Hypothesis (ii) of the previous theorem can be weakened and replaced by the following one:

(ii′) *For each* $F \in X_c$, *if* $\{F_n\}$ *is a sequence of functions* $F_n \in X_{cs}$ *such that* $|F_n(x)| \leq m(x)$, *(m integrable) and* $\tau(F_n(x), F(x)) \to 0$ *almost everywhere then* $I(F_n) \to I(F)$.

By the Lebesgue theorem and Proposition 2 we have then the following

THEOREM 2.—*If a map* I *from* X_c *into* $c(R^m)$ *satisfies* (i) *and* (iii) *of Theorem* 1 *and* (ii′) *then* $I(F) = J(F)$ *for each* $F \in X_c$.

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