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Decomposition of representations with highest weight of semisimple Lie algebra into representations of regular subalgebra

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Algebra. — *Decomposition of representations with highest weight of semisimple Lie algebra into representations of regular subalgebra.* Nota di ALESSIO U. KLIMYK, presentata (*) dal Socio G. WATAGHIN.

RIASSUNTO. — La decomposizione di una rappresentazione di peso massimo di un'algebra di Lie semisemplice, è dedotta per il caso in cui questa rappresentazione è ristretta a rappresentazioni irriducibili della rispettiva algebra (subalgebra) regolare. Il caso di rappresentazioni con numero finito di dimensioni è anche esaminato.

Some physical considerations lead to the questions: what is the decomposition of a given irreducible representation of an algebra (group) when restricted to irreducible representations of a subalgebra (subgroup). Here we deal with such a problem.

§ 1. Let G be a complex semisimple Lie algebra of rank l. It is well known that the algebra G is generated by elements $h_i, e_i, f_i, i = 1, 2, \cdots, l$ where h_i are a basis for the Cartan subalgebra G^0 of G, the elements e_i belong to the root spaces G^{α_i} of G corresponding to the simple roots α_i, $i = 1, 2, \cdots, l$, the element f_i belong to the root spaces $G^{-\alpha_i}$. Operators of a representation of G corresponding to elements h_i, e_i, f_i will be denoted by H_i, E_i, F_i.

By a representation with highest weight (or with highest vector) we mean one which satisfies the following conditions:

(*i*) a space V of a representation has such a vector x that $Hx = \Lambda(h)\, x$ for all $h \in G^0$,

(*ii*) $E_i x = 0$, $i = 1, 2, \cdots, l$,

(*iii*) a space V is generated by the collection of vectors x, $F_{i_1} F_{i_2} \cdots \cdots F_{i_r} x$, $i_j = 1, 2, \cdots, l$; $r = 1, 2, \cdots$

A linear function $\Lambda(h)$ is called the highest weight of the representation.

For example, a finite-dimensional representation of G is a representation with highest weight.

It is proved [1] that a one-to-one correspondence exists between the non-equivalent irreducible representations with highest weights of the algebra G and the complex linear functions with domain G^0, which are highest weights. Thus a complex linear function (a highest weight) uniquely characterizes a class of equivalent irreducible representations. The irreducible representation with highest weight $\Lambda(h)$ will be denoted by D_Λ.

If the highest weight $\Lambda(h)$ satisfies the condition that $\frac{2(\Lambda, \alpha_i)}{(\alpha_i, \alpha_i)}$ is a non-negative integer for every simple root α_i, $i = 1, 2, \cdots, l$, then the representation D_Λ is finite-dimensional.

(*) Nella seduta dell'8 febbraio 1969.

Everywhere in the following we shall consider only irreducible representations with highest weights $\Lambda(h)$ for which $\frac{2(\Lambda, \alpha_i)}{(\alpha_i, \alpha_i)}$ are the integers (but not only non-negative) for all simple roots α_i. Formulas were proved for the characters and weight multiplicities of such irreducible representations [2–4]. By a character χ of a weight representation [(1)] we mean a formal sum

$$\chi = \sum_{M} n_M e(M), \tag{1}$$

where the summation is over all weights of a representation, n_M is the multiplicity of the weight M, and $e(M)$ are formal exponents with multiplication operator

$$e(M')\, e(M'') = e(M' + M''). \tag{2}$$

A character of a representation with highest weight is an element of the associative algebra U consisting of elements $\beta = \Sigma a_M e(M)$, for which the following condition is satisfied: for every β linear function M_β exists such that $a_{M''}$ may be different from O if and only if $M'' = M_\beta - \sum_{i=1}^{l} m_i \alpha_i$ where m_i are non-negative integers. The algebra U is a commutative domain of integrity. Therefore, the operations we do over the characters are justified.

A character of the representation D_Λ can be written

$$\chi_\Lambda = \frac{X_\Lambda}{Q} = \frac{\sum_{S} (\det S)\, e(S(\Lambda + R))}{\sum_{S \in W} (\det S)\, e(SR)}, \tag{3}$$

where W is the Weyl group of G and summation in the numerator is made over all S, S $\in$ W, which satisfy the condition (8) in [2]. If division is fulfilled according to (2) we have the character in the form (1). A element X_Λ of the associative algebra U is called a characteristic of the representation D_Λ.

The following two problems are of interest for physics: decomposition of a tensor product of irreducible representations into irreducible representations and decomposition of a representation of a semisimple Lie algebra G into irreducible representations of a semisimple subalgebra G′ (restriction of a representation of an algebra into a subalgebra). Studying these problems we can use the characters of the representations. Some methods of such a decomposition are contained in [5]. Here we study the decomposition of the irreducible representations of the algebra G into irreducible representations of a regular subalgebra. A subalgebra G′ of G is regular if it is semisimple and the root system of G′ is a part of the root system of G.

(1) The representation is weight one if the representation space has a basis consisting of eigenvectors of operators H, $h \in G^0$.

§ 2. Let $\varphi_1, \varphi_2, \cdots, \varphi_n$ be all the positive roots of the algebra G. Let the roots be numbered so that $\varphi_1, \varphi_2, \cdots, \varphi_{n-n'}$ are the roots of the algebra G which do not belong to G′ and $\varphi_{n-n'+1}, \cdots, \varphi_n$ are all the positive roots of the subalgebra G′. Let us restrict the representation D_Λ of G onto G′. The restricted representation is decomposed into a sum of irreducible representations [2] of G′

$$D_\Lambda = \sum_\lambda \rho_\Lambda(\lambda)\, D'_\lambda,$$

where D'_λ are irreducible representations of G′ with highest weights λ, $\rho_\Lambda(\lambda)$ are the multiplicities of irreducible representations D'_λ in the representation D_Λ. For the characters of representations we have that

$$\chi_\Lambda = \sum_\lambda \rho_\Lambda(\lambda)\, \chi'_\lambda .$$

Here in the character $\chi_\Lambda = \Sigma n_M\, e(M)$ the linear functions $M(h)$ must be considered only with domain G'^0, where G'^0 is the Cartan subalgebra of G′. According to (3) relation (4) can be written

$$\frac{X_\Lambda}{Q} = \sum_\lambda \rho_\Lambda(\lambda) \frac{X'_\lambda}{Q'},$$

where X'_λ and Q' are for G′ the same as X_Λ and Q are for G. The denominators of (5) can be represented in the form

$$Q = \sum_{S \in W} (\det S)\, e(SR) = \prod_{k=1}^{n} \left(e\left(\frac{\varphi_k}{2}\right) - e\left(-\frac{\varphi_k}{2}\right)\right) = \tag{6}$$

$$= e(R) \prod_{k-1}^{n} (1 - e(-\varphi_k)),$$

$$Q' = e(R') \prod_{k=n-n'+1}^{n} (1 - e(-\varphi_k)). \tag{7}$$

If we multiply relation (5) with Q' and apply (6) and (7), we get

$$\frac{X_\Lambda}{e(R-R') \prod_{k=1}^{n-n'} (1 - e(-\varphi_k))} = \sum_\lambda \rho_\Lambda(\lambda)\, X'_\lambda . \tag{8}$$

Every term $(1 - e(-\varphi_k))^{-1}$ is an element of the associative algebra U and

$$(1 - e(-\varphi_k))^{-1} = 1 + e(-\varphi_k) + e(-2\varphi_k) + \cdots$$

One obtains that

$$\prod_{k=1}^{n-n'} (1 - e(-\varphi_k))^{-1} = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_{n-n'}=0}^{\infty} e\left(-\sum_{k=1}^{n-n'} i_k \varphi_k\right). \tag{9}$$

(2) It must be remarked that the obtained sum can be semidirect, that is, the restricted representation can be reducible but not completely reducible.

Using (9) and formula for X_Λ now relation (8) can be written

$$\sum_{\lambda} \rho_\Lambda(\lambda) X'_\lambda = \tag{10}$$

$$= \sum_{S} (\det S) \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_{n-n'}=0}^{\infty} e\left(S(\bar{\Lambda} + \bar{R}) + R' - \bar{R} - \sum_{i=1}^{n-n'} i_k \bar{\varphi}_k\right).$$

In relation (10) all linear functions must be considered only with domain G'^0. In order to emphasize this we wrote (10) with bars over the corresponding functions.

The right hand side of (10) allows to find the characteristics X'_λ. Once the characteristics are found, we shall know the corresponding irreducible representations. In the case of finite-dimensional representations

$$X'_\lambda = \sum_{S' \in W'} (\det S')\, e\,(S'(\lambda + R'))$$

and there is only a single exponent, namely $e(\lambda + R')$, for which the corresponding linear function $\lambda + R'$ is dominant. Hence if the representation D_Λ is finite-dimensional, it is not necessary to find all the sum on the right hand side of (10). It is sufficient to find only the summands for which the corresponding linear functions are dominant. In the case of a infinite-dimensional representation D_Λ it is necessary to know all the sum.

In fact the finding of the sum on the right hand side of (10) is a geometrical operation (3). The linear functions with domain G'^0 are considered as points of a coordinate space [6]. The right hand side of (10) is found in the following manner. We successively add the vector $-\bar{\varphi}_1$ to the point $S(\bar{\Lambda} + \bar{R}) + R' - \bar{R}$. The obtained collection of points lies on a single straight line. These points are successively shifted by the vector $-\bar{\varphi}_2$, and we obtain a lattice in the plane. This lattice is successively shifted by the vector $-\bar{\varphi}_3$. If the vector $-\bar{\varphi}_3$ lies on the plane of vectors $-\bar{\varphi}_1$, and $-\bar{\varphi}_2$ we have the superposition of lattices. The superposition increases the multiplicities of points. We must do such a geometrical operation with the roots $-\bar{\varphi}_4, -\bar{\varphi}_5, \cdots, -\bar{\varphi}_{n-n'}$. One must remember that if under a shift the points coincide, their multiplicities are added. The obtained lattice is a geometrical representation of the sum

$$\sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_{n-n'}=0}^{\infty} e\left(S(\bar{\Lambda} + \bar{R}) + R' - \bar{R} - \sum_{k=1}^{n-n'} i_k \bar{\varphi}_k\right).$$

In order to obtain a geometrical representation of the whole the right hand side of (10) we must multiply the multiplicities of the points in the lattice by det S, find such a lattice for every S satisfying the condition (8) in [2] and add the multiplicities of coinciding points of all the lattices.

(3) A similar geometrical situation can be found in [6, 7]. For the detailed presentation of the finding of such a sum we refer the reader to these articles.

§ 3. Using formula (10) we prove the relation for $\rho_\Lambda(\lambda)$. We represent a multiplicity $\rho_\Lambda(\lambda)$ as a sum of terms $P^{(\Delta'_+)}(\mu)$. The collection of all the positive roots of the subalgebra G', that is, the collection of the roots $\varphi_{n-n'+1}, \cdots, \varphi_n$ is denoted by Δ'_+. $P^{(\Delta'_+)}(\mu)$ equals the number of ways in which the linear function μ with domain G'^0 can be partitioned into a sum of positive roots $\bar\varphi_1, \bar\varphi_2, \cdots, \bar\varphi_{n-n'}$ on the algebra G which do not belong to Δ'_+. Using the term $P^{(\Delta'_+)}(\mu)$ the formula (10) can be written as

$$\sum_\lambda \rho_\Lambda(\lambda)\, X'_\lambda = \sum_S (\det S) \sum_\mu P^{(\Delta'_+)}(\mu)\, e(S(\bar\Lambda + \bar R) + R' - \bar R - \mu), \tag{11}$$

where the second sum of the right hand side is made over all the linear functions μ, which can be represented as $\sum_{i=1}^{n-n'} i_k \bar\varphi_k$, $i_k \geq 0$, $k = 1, 2, \cdots, n - n'$. Suppose that the representation D_Λ is finite-dimensional. The formula (11) then is

$$\sum_\lambda \rho_\Lambda(\lambda) \sum_{S' \in W'} (\det S')\, e(S'(\lambda + R')) =$$

$$\sum_{S \in W} (\det S) \sum_\mu P^{(\Delta'_+)}(\mu)\, e(S(\bar\Lambda + \bar R) + R' - \bar R - \mu).$$

Let us consider the coefficients under fixed $e(\lambda + R')$ where λ is dominant. It is easy to see that the left hand side has only a single term with fixed $e(\lambda + R')$ and we obtain

$$\rho_\Lambda(\lambda) = \sum_{S \in W} (\det S)\, P^{(\Delta'_+)}\, e(S(\bar\Lambda + \bar R) + R' - \bar R - \lambda).$$

It is a formula for $\rho_\Lambda(\lambda)$ for the case of a finite-dimensional representation D_Λ. A formula for $\rho_\Lambda(\lambda)$ in the case of the restriction of a finite-dimensional representation of a semisimple Lie algebra onto a regular subalgebra of maximal rank was proved by Mandelzweig [8].

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