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Flows of heat

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Analisi matematica. — *Flows of heat.* Nota di JOSEF KRÁL, presentata (*) dal Corrisp. G. FICHERA.

RIASSUNTO. — Sia D un insieme aperto con la frontiera compatta $B \neq \emptyset$ nello spazio Euclideo R^m, $m > 1$. Fissiamo $T_1 < T_2$, poniamo $F = B \times \langle T_1, T_2 \rangle$, $E = D \times (T_1, T_2)$ e indichiamo con $\mathfrak{D}_{T_2}$ l'insieme delle funzioni $C^\infty(R^{m+1})$ a supporto compatto contenuto in $R^m \times (-\infty, T_2)$. Per ogni misura di Borel μ a supporto contenuto in F consideriamo il corrispondente potenziale del calore $U\mu$ e definiamo su $\mathfrak{D}_{T_2}$ il funzionale $H\mu$ come

$$\langle \varphi, H\mu \rangle = \int_E \left[\sum_{j=1}^m \frac{\partial U\mu(z)}{\partial z_j} \cdot \frac{\partial \varphi(z)}{\partial z_j} - U\mu(z) \frac{\partial \varphi(z)}{\partial z_{m+1}} \right] dz, \ \varphi \in \mathfrak{D}_{T_2}.$$

In questa Nota sono studiate le proprietà dell'operatore $H: \mu \to H\mu$ e le sue applicazioni al problema di Fourier.

We shall deal with potentials associated with the well-known kernel

$$G(z_1, \cdots, z_{m+1}) = z_{m+1}^{-(1/2)m} \exp\left(-\sum_{j=1}^m z_j^2/4 z_{m+1}\right), \ z_{m+1} > 0,$$

$$G(z_1, \cdots, z_{m+1}) = 0 \quad , \quad z_{m+1} \leq 0.$$

Let D be an open set with a compact boundary $B \neq \emptyset$ in the Euclidean m-space R^m, $m > 1$, and put

$$C = B \times \langle T_1, T_2 \rangle \quad , \quad E = D \times (T_1, T_2),$$

where $T_1 < T_2$ are fixed real numbers. Let $\mathfrak{B}'$ denote the class of all finite signed Borel measures μ in R^{m+1} such that $|\mu|(R^{m+1} \setminus C) = 0$, where $|\mu|(M)$ denotes the variation of μ on $M \subset R^{m+1}$. $\mathfrak{B}'$ will be considered as a Banach space with the norm $\|\mu\| = |\mu|(C)$, $\mu \in \mathfrak{B}'$. It is easy to verify that, for each $\mu \in \mathfrak{B}'$, the corresponding thermal potential

$$U\mu(z) = \int_C G(z - \zeta) \, d\mu(\zeta)$$

has integrable derivatives $\frac{\partial U\mu(z)}{\partial z_j}$ $(j = 1, \cdots, m)$ on E. This permits us to introduce the distribution $H\mu$ in the half-space $R_{T_2} = R^m \times (-\infty, T_2)$ defining for $\varphi \in \mathfrak{D}_{T_2}$ (= the class of all infinitely differentiable functions in R^{m+1} with compact support in R_{T_2})

$$\langle \varphi, H\mu \rangle = \int_E \left[\sum_{j=1}^m \frac{\partial U\mu(z)}{\partial z_j} \cdot \frac{\partial \varphi(z)}{\partial z_j} - U\mu(z) \frac{\partial \varphi(z)}{\partial z_{m+1}} \right] dz.$$

(*) Nella seduta dell'8 febbraio 1969.

$H\mu$ will be termed the heat flow associated with μ and E. We shall say that $H\mu$ is a measure provided there is a $\nu \in \mathfrak{B}'$ such that

$$\langle \varphi , H\mu \rangle = \int_C \varphi \, d\nu \, , \; \varphi \in \mathfrak{D}_{T_2} \, .$$

If this is the case then ν (which is uniquely determined) will be identified with $H\mu$. Our main objective is to determine necessary and sufficient geometric conditions on D guaranteeing that $H\mu$ is a measure for each $\mu \in \mathfrak{B}'$ and to investigate relations between analytical properties of $\dot{H}$ and geometrical properties of D. For this purpose it is useful to adopt the following concept of a hit as introduced in (1), definition 1.5. Given $x \in R^m$, $r > 0$ and $\theta \in \Gamma = R^m \cap \{\theta ; |\theta| = 1\}$, let $S_r(\theta , x) = \{x + \rho\theta ; 0 < \rho < r\}$. A point $y \in S_r(\theta , x)$ will be termed a hit of $S_r(\theta , x)$ on D provided each neighborhood of y in $S_r(\theta , x)$ meets both $D \cap S_r(\theta , x)$ and $S_r(\theta , x) \setminus D$ in a set of positive linear measure. If $n_r(\theta , x)$ denotes the number (possibly 0 or ∞) of all the hits of $S_r(\theta , x)$ on D then, by proposition 1.6 in (1), $n_r(\theta , x)$ is a Baire function of the variable $\theta \in \Gamma$ and we are justified in defining

$$v_r(x) = \int_\Gamma n_r(\theta , x) \, d\sigma_\Gamma(\theta),$$

where $d\sigma_\Gamma$ is the area element on Γ. Arguments similar to those used for the proof of theorem 1.13 in (1) permit one to establish the following result.

THEOREM 1.—*In order that* $H\mu$ *be a measure for each* $\mu \in \mathfrak{B}'$ *it is necessary and sufficient that*

$$\sup_{x \in B} v_r(x) < \infty \, . \tag{1}$$

If the condition (1) *is fulfilled then* $H : \mu \to H\mu$ *is a bounded operator on* $\mathfrak{B}'$.

In what follows we always assume (1). By lemma 2.7 in (1), this implies the existence of the density

$$d_D(x) = \lim_{r \to 0+} \frac{\text{volume}\,(D \cap \{y ; |x - y| < r\})}{\text{volume}\,(\{y ; |x - y| < r\})}$$

for any $x \in R^m$. Let $\mathfrak{B}$ denote the Banach space of all continuous functions on $F = B \times \langle T_1 , T_2 \rangle$ vanishing on $F \setminus C = B \times \{T_2\}$, equipped with the supremum norm. For further investigation of H it is useful to have an integral representation for the operator W_0 on $\mathfrak{B}$ whose dual is H. Given $x \in B$, $\theta \in \Gamma$ and $r > 0$, we let $s(r ; x , \theta) = \varepsilon \, (= \pm 1)$ if there is a $\delta > 0$ such that $x + (r + \varepsilon\rho)\,\theta \in D$, $x + (r - \varepsilon\rho)\,\theta \in R^m \setminus D$ for almost every $\rho \in (0 , \delta)$; otherwise we set $s(r ; x , \theta) = 0$ (compare (1), 2.4). If $f \in \mathfrak{B}$, $\eta > 0$

(1) JOSEF KRÁL. *The Fredholm method in potential theory*, « Transactions of the American Mathematical Society », *125*, 511–547 (1966).

and $z = [x, t] \in F$, we put for $\theta \in \Gamma$

$$\Sigma_f(z; \eta, \theta) = \sum_r f\left(x + r\theta, t + \frac{r^2}{4\eta}\right) s(r; x, \theta),$$

the sum on the right-hand side being extended over r satisfying $0 < r < 2[\eta(T_2 - t)]^{1/2}$. Then $\Sigma_f(z; \eta, \theta)$ is defined for σ_Γ–almost every $\theta \in \Gamma$ and the integral

$$Wf(z) = \int_0^\infty d\eta \int_\Gamma e^{-\eta} \eta^{1/2m} \Sigma_f(z; \eta, \theta) \, d\sigma_\Gamma(\theta)$$

is meaningful for all $z \in F$ and $f \in \mathfrak{B}$. We have now the following analogue of lemma 3.4 in (1).

PROPOSITION 1.—*Given* $z = [z_1, \cdots, z_{m+1}] \in F$, let $\hat{z} = [z_1, \cdots, z_m]$ *and define for* $f \in \mathfrak{B}$

$$W_0 f(z) = 2^{m-1}[Wf(z) + 2\pi^{1/2m} d_D(\hat{z}) f(z)].$$

Then $W_0 f \in \mathfrak{B}$ *for each* $f \in \mathfrak{B}$, *the operator* $W_0 : f \to W_0 f$ *is bounded on* $\mathfrak{B}$ *and* H *is dual to* W_0.

Consider now the operators

$$W_\alpha = W_0 - 2^m \pi^{1/2m} \alpha I \quad , \quad \alpha \in R^1,$$

where I is the identity operator on $\mathfrak{B}$. Since $H = (2^m \pi^{1/2m} \alpha I + W_\alpha)'$ [here $(\ldots)'$ stands for the dual of $(\ldots)$], it is important to evaluate

$$\omega W_\alpha = \inf_T \| W_\alpha - T \|$$

with T ranging over all compact operators acting on $\mathfrak{B}$. It appears that conditions guaranteeing the validity of the estimate

$$\inf_{\alpha \neq 0} \frac{\omega W_\alpha}{|\alpha|} < 2^m \pi^{1/2m} \tag{2}$$

may be expressed in terms of the quantities $V_0(M)$ defined for $M \subset B$ as follows:

$$V_0(\varnothing) = 0,$$

$$V_0(M) = \lim_{r \to 0+} \sup_{x \in M} v_r(x), M \neq \varnothing.$$

They are given in the following theorem whose proof is based on proposition 1 and reasonings similar to those used in the proof of theorem 3.6 in (1).

THEOREM 2.—*Let* $A = 2\pi^{1/2m}/\Gamma\left(\frac{1}{2}m\right)$ *and put*

$$B_1 = B \cap \{x; d_D(x) = 1\}, B_2 = B \cap \left\{x; d_D(x) = \frac{1}{2}\right\}.$$

Then (2) *is valid if and only if*

$$V_0(B_1) < A \quad \text{and} \quad V_0(B_2) < \frac{1}{2} A. \tag{3}$$

If the conditions (3) *are fulfilled then* $\gamma \in R^1 \setminus \{0\}$ *satisfying*

$$\frac{\omega W_\gamma}{|\gamma|} = \inf_{\alpha \neq 0} \frac{\omega W_\alpha}{|\alpha|}$$

is uniquely determined and one may distinguish the following cases (*i*)–(*iii*):

(*i*) $B_1 = \varnothing$,

(*ii*) $B_2 = \varnothing$ or $V_0(B_1) \geq V_0(B_2) + \frac{1}{2} A$,

(*iii*) $B_1 \neq \varnothing \neq B_2$ and $|V_0(B_1) - V_0(B_2)| \leq \frac{1}{2} A$.

The corresponding values of γ *and* $a = \frac{\omega W_\gamma}{2^m |\gamma| \pi^{1/2m}}$ *are then given as follows*:

(*i*) $\Rightarrow \gamma = \frac{1}{2}$, $\gamma A a = V_0(B_2)$,

(*ii*) $\Rightarrow \gamma = 1$, $A a = V_0(B_1)$,

(*iii*) $\Rightarrow \gamma = \frac{3}{4} + \frac{V_0(B_1) - V_0(B_2)}{2A}$, $a = \frac{V_0(B_1) + V_0(B_2) + \frac{1}{2} A}{V_0(B_1) - V_0(B_2) + \frac{3}{2} A}$.

Combining the above results with the Riesz-Schauder theory one obtains the following corollary concerning the Fourier problem.

COROLLARY.—*If* D *fulfils* (3) *then* H *has a bounded inverse on* $\mathfrak{B}'$.

Complete proofs together with further references will be given elsewhere.