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On some classes of normaloid operators

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RENDICONTI

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Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi funzionale. — *On some classes of normaloid operators.* Nota di GHEORGHE CONSTANTIN, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota si dà una generalizzazione del teorema di Arens di localizzazione dello spettro per una classe di operatori normaloide e un teorema di struttura per gli operatori invariantemente normaloide.

1. Let T be a bounded linear operator on a Hilbert space H.

The operator T is called normaloid if

$$\sup_{\| x \| = 1} \{ | < Tx , x > | \} = \| T \|$$

The purpose of this Note is to obtain some information about normaloid operators. In the first part of this Note an extension of theorem of Arens about location of spectra is obtained. In the second part some structure theorems for Riesz operators which are in some subclass of normaloid operators are given.

2. Let T be an operator on Hilbert space H. Following [3] the operator T is of class (N) if

$$\| Tx \|^2 \leq \| T^2 x \|$$

for every $\| x \| = 1$.

In what follows, using ideas of [1], we consider the problem of location of spectrum for a class of operators defined below: for every complex number λ, $T - \lambda$ is of class (N).

(*) Nella seduta dell'8 febbraio 1969.

We need the following

LEMMA 1.1.—*The operator* T *is normaloid.*

Proof.—See [3].

LEMMA 1.2.—*If* T^{-1} *exists then it is also of class* (N).

Proof.—See [4].

With these lemmas we prove.

THEOREM 1.1.—*If* $T = A + iB$ *then every* $\lambda = x + iy$ *in the spectrum satisfies the condition*

$$| x \cos \theta + y \sin \theta | \leq \| A \cos \theta + B \sin \theta \|$$

with $0 \leq \theta \leq 2\pi$.

Proof.—The proof uses the ideas of Arens theorem [1].

Let λ_0, $|\lambda_0| > \|T\|$. Then $T - \lambda_0$ has an inverse and is of class (N) and thus normaloid. It is clear now that there exists $\mu_0 \in \sigma(T)$ such that

$$\|(T - \lambda_0)\| = |\mu_0 - \lambda_0|.$$

Since for small λ

$$(T - \lambda_0 - \lambda)^{-1} = (T - \lambda_0)^{-1} \sum_0^\infty \lambda^n (T - \lambda_0)^{-n}$$

and if $g = (T - \lambda_0)(T^* - \bar{\lambda}_0)$ we have

$$\|(T - \lambda_0)^{-n}\| = \|(T^* - \bar{\lambda}_0)^{-n}\|$$

and

$$R^{-1} = \|(T - \lambda_0)^{-1}\| = \|(T^* - \bar{\lambda}_0)^{-1}\| = \|g^{-1}\|^{1/2}.$$

But

$$T - \lambda_0 = A - \rho \cos \theta + i (B - \rho \sin \theta)$$

$$T^* - \bar{\lambda}_0 = A - \rho \cos \theta - i (B - \rho \sin \theta)$$

and

$$g = \rho^2 (1 - \omega) = \rho^2 \left[1 - \frac{2(A \cos \theta + B \sin \theta)}{\rho} + \frac{TT^*}{\rho^2} \right]$$

which gives

$$\frac{1}{R} \leq \frac{1}{\rho (1 - \|\omega\|)^{1/2}}.$$

Thus, if $T - \lambda - \lambda_0$ has no inverse, $\lambda_0 + \lambda = x + iy$ is in the spectrum and thus

$$|\lambda| > R$$

which implies for $\rho \to \infty$

$$x \cos \theta + y \sin \theta \leq \| A \cos \theta + B \sin \theta \|$$

and for $\theta + \pi$,

$$-x \cos \theta - y \sin \theta \leq \| A \cos \theta + B \sin \theta \|$$

and the theorem is proved.

COROLLARY.—*If* T *has real spectrum then* T *is selfadjoint.*

3. In this section we prove some theorems on operators of Riesz type. The concept of an operator of Riesz type was introduced by A. F. Ruston by using as an axiomatic system those properties of compact operators used by F. Riesz in his original discussion of integral equations.

If M, N are closed subspaces of H invariant under T such that H is the direct sum $M \oplus N$, the pair (M, N) is said to reduce T.

A Riesz point of $\sigma(T)$ is a point λ of $\sigma(T)$ such that

$$H = N(\lambda ; T) \oplus F(\lambda ; T)$$

where $\dim N < \infty$ and F is closed, the pair (N, F) reduces T, and $\lambda - T$ restricted to N is nilpotent while $\lambda - T$ restricted to F is a homeomorphism.

DEFINITION. 2.1.—*T is a Riesz operator on H if each point of* $\sigma(T) - \{0\}$ *is a Riesz point.*

A subclass of normaloid operators—the class of invariant normaloid operators—is given by

DEFINITION 2.2 [5].—*An operator T is invariant normaloid if the restriction to every invariant subspace is also normaloid.*

This class contains operators of class (N) (see [5]) and also the class of hyponormal operators [3].

PROPOSITION 2.1.—*If T is a Riesz invariant normaloid operator, then T is normal.*

Proof.—Since T is a Riesz operator and normaloid there is a $\lambda \in \sigma(T)$ such that $|\lambda| = \|T\|$ and $\eta_T(\lambda) = \{x : Tx = \lambda x\} \neq \{0\}$.

Thus $\bar{\lambda} \in \sigma(T^*)$ since we have

$$\|T^* x - \bar{\lambda} x\|^2 = \|T^* x\|^2 - \bar{\lambda} < x , T^* x > - \lambda < T^* x , x > + |\lambda^2| \, \|x\|^2 \leq 0$$

for every $x \in \eta_T(\lambda)$ and therefore $T/\eta_T(\lambda)$, the restriction of T onto $\eta_T(\lambda)$ is normal.

Since T is invariant normaloid and $\eta_T(\lambda)$ reduces T (this means that $\eta_T(\lambda)$ is invariant under T and T*), if we denote $M = \eta_T^{\perp}(\lambda)$ it follows that T/M is a Riesz normaloid operator. If we continue in this way we obtain that $\{\eta_T(\lambda) : \lambda \in \sigma_p(T)\}$ is a mutually orthogonal family which reduces T. To complete the proof of the proposition we have only to prove that the restriction T_1 of T onto $H_1 = H_0^{\perp}$ is zero. Suppose the contrary. Then T_1 is a non-zero Riesz normaloid operator and it follows that there exists a point $\mu \in \sigma(T_1)$ such as $|\mu| = \|T_1\|$. Hence $T_1 x = \mu x$ for some non-zero vector $x \in H_1$. This is a contradiction which implies $T_1 = 0$ and T normal.

Remark.—If T is an invariant normaloid operator such that $T^n T^{*n}$ is a Riesz operator for some integer $n \geq 0$ then T is normal.

The proof follows from Theorem 3 [5] since $T^n T^{*n}$ is a self-adjoint Riesz operator and thus it is a compact operator.

PROPOSITION 2.2.—*If* A *and* B *are bounded linear operators on* H *which commutes and* A — B *is a Riesz operator then they have the same spectrum with except the eigenvalues.*

Proof.—We have to prove that if $\lambda \in \sigma(A) - \sigma_p(A)$ then $\lambda \in \sigma(B)$. Because this property is invariant to a translation with λ, we can consider the case $\lambda = 0$. If $0 \in \sigma(A) - \sigma_p(A)$ then A is not an invertible operator but the null space N(A) coincides with $\{0\}$ and we prove that B is not invertible.

Suppose the contrary and let

$$A = B + (A - B) = B(1 - B^{-1}(A - B)). \tag{1}$$

Since $AB = BA$ it follows that $B^{-1}A = AB^{-1}$ and

$$B^{-1}(A - B) = (A - B)B^{-1}$$

Since $A - B$ is a Riesz operator and B^{-1} is a bounded operator it follows that $B^{-1}(A + B)$ is a Riesz operator and by relation (1) it follows that -1 is an eigenvalue for $B^{-1}(A - B)$ and thus $N(A) \neq \{0\}$ or $1 + B^{-1}(A - B)$ is invertible but this implies that A is invertible which is a contradiction and we conclude that $0 \in \sigma(B)$.

REFERENCE.

[1] R. ARENS, *Representations of *–algebras*, « Duke Math. J. », *14* (2), 269–282 (1947).

[2] P. R. HALMOS, *A Hilbert space problem book*, Van Nostrand, (1967).

[3] V. ISTRĂŢESCU, *On some hyponormal operators*, « Pacific J. Math. », *22* (3), 413–417 (1967).

[4] V. ISTRĂŢESCU, T. SAITO and T. YOSHINO. *On class of operators*, « Tohoku Math. J. », *18* (4), 410–414 (1966).

[5] V. ISTRĂŢESCU and I. ISTRĂŢESCU, *On normaloid operators*, « Math. Zeitschr. », *105*, 153–156 (1968).

[6] C. R. PUTNAM, *On the spectra of semi–normal operators*, « Trans. Amer. Math. Soc. », *119* (3), 509–524 (1965).

[7] A. F. RUSTON, *Operators with Fredholm theory*, « J. London Math. Soc. », *29*, 318–326 (1954).

[8] T. WEST, *Riesz operators in Banach spaces*, « Proc. London Math. Soc. », *16* (3), 131–140 (1966).