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Uniqueness and almost-periodicity theorems for a non linear wave equation

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RENDICONTI

DELLE SEDUTE

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Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi matematica. — *Uniqueness and almost-periodicity theorems for a non linear wave equation* (*). Nota di LUIGI AMERIO e GIOVANNI PROUSE, presentata (**) dal Corrisp. LUIGI AMERIO.

RIASSUNTO. — Si dimostrano un teorema di unicità di soluzioni limitate e un teorema di quasi periodicità, concernenti l'equazione delle onde con termine dissipativo funzione discontinua della velocità.

1\. – In the present paper, closely related to [1], we consider the following non linear wave equation:

$$A(x)\, y(t,x) - y_{tt}(t,x) + f(t,x) = \beta(y_t(t,x)) \tag{1.1}$$

being $x \in \Omega$ (open, bounded and connected set $\subset R^m$), $t \in J = -\infty \frown +\infty$. All functions are supposed to be real and the derivatives are taken in the sense of distributions.

Setting

$$A(x) = \sum_{j,k}^{1\cdots m} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial}{\partial x_k}\right) - a_0(x), \tag{1.2}$$

we assume, as usual, that

$$\begin{gathered} a_{jk}(x)\ ,\ a_0(x) \in L^\infty(\Omega)\,; \qquad a_{jk}(x) = a_{kj}(x)\ ;\ a_0(x) \geq 0\,; \\ \sum_{j,k}^{1\cdots m} a_{jk}(x)\,\xi_j\,\xi_k \geq \nu \sum_{1}^{m}{}_j\, \xi_j^2 \qquad \forall(\xi_1,\cdots,\xi_m) \in R^m \quad (\nu > 0)\,. \end{gathered} \tag{1.3}$$

(*) Istituto Matematico del Politecnico di Milano. Gruppo di ricerca n. 12 del Comitato per la Matematica del C.N.R.

(**) Nella seduta del 19 novembre 1968.

Moreover $\beta(\eta)$ is supposed to be *a non decreasing* function of the variable $\eta \in a \overline{\quad} b$, $a < 0 < b$, such that $\beta(0^-) \leq 0 \leq \beta(0^+)$ and

$$(1.4) \qquad \beta(a^+) = -\infty \quad \textit{if} \quad a > -\infty \quad , \quad \beta(b^-) = +\infty \quad \textit{if} \quad b < +\infty .$$

We shall denote by $\{\eta_s\}$ the sequence of points at which $\beta(\eta)$ has discontinuities.

Setting $y(t) = \{y(t, x) ; x \in \Omega\}$, $y'(t) = \{y_t(t, x) ; x \in \Omega\}$, $y''(t) = \{y_{tt}(t, x) ; x \in \Omega\}$, $Ay(t) = \{A(x) y(t, x) ; x \in \Omega\}$, $f(t) = \{f(t, x) ; x \in \Omega\}$, $\beta(y'(t)) = \{\beta(y_t(t, x)) ; x \in \Omega\}$, we can also write equation (1.1) in the operational form

$$(1.5) \qquad Ay(t) - y''(t) + f(t) = \beta(y'(t)) .$$

Let E be the *energy* space ($E = H_0^1 \times L^2$, where $H_0^1 = H_0^1(\Omega)$, $L^2 = L^2(\Omega)$) and

$$\|y(t)\|_{H_0^1} = \left\{ \int_\Omega \left(\sum_{j,k}^{1\cdots m} a_{jk}(x) \frac{\partial y(t,x)}{\partial x_j} \frac{\partial y(t,x)}{\partial x_k} + a_0(x)\, y^2(t,x) \right) dx \right\}^{1/2} ,$$

$$\|y(t)\|_E = \{\|y(t)\|_{H_0^1}^2 + \|y'(t)\|_{L^2}^2\}^{1/2} .$$

Following the definition given in [1], we shall say that $y(t)$ is *a solution of* (1.5) *on the interval* $J_0 = t_0 \vdash\!\!\!\overline{\quad} +\infty$, *satisfying the boundary condition*

$$(1.6) \qquad y(t, x)\big|_{x \in \partial\Omega} = 0 \qquad (t \in J_0),$$

if:

1) $y(t), y'(t) \in C^0(J_0 ; H_0^1)$; $y''(t), Ay(t) \in L^\infty_{loc}(J_0 ; L^2)$;

2) *it results*, almost-everywhere (a.e.) on $Q = J_0 \times \Omega$,

$$(1.7) \qquad a < y_t(t, x) < b ,$$

$$(1.8) \qquad A(x)\, y(t, x) - y_{tt}(t, x) + f(t, x) \in \beta((y_t(t, x))^-) \vdash\!\!\dashv \beta((y_t(t, x))^+) .$$

Setting $\bar\beta(0) = 0$, $\bar\beta(\eta) = \beta(\eta^-)$ on $0 \overline{\quad} b$, $\bar\beta(\eta) = \beta(\eta^+)$ on $a \overline{\quad} 0$, *assume that*: $u_0 \in H_0^1$, $Au_0 \in L^2$; $u_1 \in H_0^1$, $u_1(x) \in a \overline{\quad} b$ a.e. on Ω, $\bar\beta(u_1) \in L^2$; $f(t_0) \in L^2$, $f'(t) \in L^1_{loc}(J_0 ; L^2)$. *There exists then* (as has been proved in [1]) *one and only one solution*, $y(t)$, *of problem* (1.5), (1.6) *such that* $y(t_0) = u_0$, $y'(t_0) = u_1$.

We shall say, moreover, that $y(t)$, *defined on all* J, *is a solution of problem* (1.5), (1.6) *if conditions* 1) *and* 2), *in which we substitute* J_0 *by* J, *hold.*

In the present paper we prove, at § 2, a uniqueness theorem for solutions *bounded for* $t \to -\infty$: we define such *boundedness* in the following sense:

$$(1.9) \qquad \max \lim_{t \to -\infty} \left\{ \int_{-1}^{0} (\|Ay(t+\eta)\|_{L^2}^2 + \|y'(t+\eta)\|_E^2)\, d\eta \right\}^{1/2} < +\infty .$$

In an analogous way we define *boundedness for* $t \to +\infty$ and, consequently, *boundedness on* J.

We show afterwards, at § 3, utilizing this theorem, that *if* $f(t)$ *is a. p. and if there exists a solution* $y(t)$, *bounded on* J, *then* $y(t)$ *is a. p.*

Let us prove now that, *if* (1.9) *holds, then* $y(t)$ *results* E–*uniformly continuous and has a* E–*relatively compact range* $\mathcal{R}_{y(t)}$ *on every interval* $-\infty < t \leq \bar{t}$.

The same properties hold, moreover, on J, *if* $y(t)$ *is bounded on* J.

$y(t)$ is E–u.c. since it is, for $0 < \tau \leq 1$, $t \leq \bar{t}$, by (1.9),

$$\| y(t) - y(t-\tau) \|_E \leq \int_{t-\tau}^{t} \| y'(\eta) \|_E \, d\eta \leq \bar{M} \tau^{1/2}, \tag{1.10}$$

where $\bar{M}^2$ denotes the supremum of the integral, in (1.9), for $-\infty < t \leq \bar{t}$.

Let us prove that $\mathcal{R}_{y(t)}$, $t \leq \bar{t}$, is E–r.c. If not, there would exist $\rho > 0$ and a sequence $\{t_n\}$, $t_n \leq \bar{t}$, such that

$$\| y(t_j) - y(t_k) \|_E \geq \rho \qquad (j \neq k). \tag{1.11}$$

Taken δ, $0 < \delta \leq 1$, such that $\bar{M}\delta^{1/2} \leq \frac{\rho}{4}$, let us consider the interval $(t_n - \delta)^{|-|} t_n$. Since

$$\left\{ \int_{t_n-\delta}^{t_n} (\| Ay(\eta) \|_{L^2}^2 + \| y'(\eta) \|_E^2) \, d\eta \right\}^{1/2} \leq \bar{M},$$

there exists $\eta_n \in (t_n - \delta)^{|-|} t_n$ such that

$$\{ \| Ay(\eta_n) \|_{L^2}^2 + \| y'(\eta_n) \|_E^2 \}^{1/2} \leq \bar{M} \delta^{-1/2}.$$

Hence $\{y(\eta_n)\}$ and $\{y'(\eta_n)\}$ are, respectively, H_0^1 and L^2–r.c. sequences. Therefore we can select a subsequence (that we shall denote again by $\{y(\eta_n)\}$) such that

$$\| y(\eta_j) - y(\eta_k) \|_E \leq \frac{\rho}{4} \cdot \tag{1.12}$$

It follows, by (1.11), (1.12) and since $\bar{M}\delta^{1/2} \leq \frac{\rho}{4}$,

$$\| y(t_j) - y(t_k) \|_E \leq \| y(t_j) - y(\eta_j) \|_E + \| y(\eta_j) - y(\eta_k) \|_E +$$

$$+ \| y(\eta_k) - y(t_k) \|_E \leq \frac{3}{4} \rho,$$

which is absurd.

2. – I. (*Uniqueness theorem*). *Assume that* $\beta(\eta)$ *is strictly increasing on* $a \overline{} b$ *and continuous at the point* $\eta = 0$.

Then, if

$$\max \lim_{t \to -\infty} \left\{ \int_{-1}^{0} \| f(t+\eta) \|_{L^2}^2 \, d\eta \right\}^{1/2} = K_f < +\infty, \tag{2.1}$$

there exists at most one solution, $y(t)$, *bounded for* $t \to -\infty$.

Let $z(t)$ be another solution, bounded for $t \to -\infty$, and consider, on every interval $\Delta_p = -p \,|\!\!-\!\!|\, p$ $(p = 1, 2, \cdots)$, the sequences $\{y(t-n)\}$, $\{z(t-n)\}$, where $n = 0, 1, \cdots$

As $y(t)$ and $z(t)$ are E–u.c. and have E–r.c. ranges on the interval $-\infty \,-\!\!|\, p$, it is possible, by the (vectorial) theorem of Ascoli-Arzelà, to select a subsequence $\{n_j\} \subset \{n\}$ such that

$$\lim_{j\to\infty} y(t-n_j) \underset{E}{=} \tilde{y}(t) \quad , \quad \lim_{j\to\infty} z(t-n_j) \underset{E}{=} \tilde{z}(t) \tag{2.2}$$

uniformly on Δ_p, $\forall p$. By (1.9), (2.1) we assume, moreover, that it is, $\forall p$,

$$\begin{aligned} &\lim^*_{j\to\infty} y'(t-n_j) \underset{L^2(\Delta_p;E)}{=} \tilde{y}'(t) \quad , \quad \lim^*_{j\to\infty} z'(t-n_j) \underset{L^2(\Delta_p;E)}{=} \tilde{z}'(t) , \\ &\lim^*_{j\to\infty} Ay(t-n_j) \underset{L^2(\Delta_p;L^2)}{=} A\tilde{y}(t) \quad , \quad \lim^*_{j\to\infty} Az(t-n_j) \underset{L^2(\Delta_p;L^2)}{=} A\tilde{z}(t) \\ &\lim^*_{j\to\infty} f(t-n_j) \underset{L^2(\Delta_p;L^2)}{=} g(t) , \end{aligned} \tag{2.3}$$

where, by (2.1), (1.9),

$$\begin{aligned} &\operatorname*{Sup}_J \left\{ \int_{-1}^{0} \| g(t+\eta) \|^2_{L^2} \, d\eta \right\}^{1/2} \le K_f , \\ &\operatorname*{Sup}_J \left\{ \int_{-1}^{0} (\| A\tilde{y}(t+\eta) \|^2_{L^2} + \| \tilde{y}'(t+\eta) \|^2_E) \, d\eta \right\}^{1/2} < +\infty \end{aligned} \tag{2.4}$$

and analogously for $\tilde{z}(t)$. Setting $Q_p = \Delta_p \times \Omega$, we deduce moreover from (2.2), $\forall p$,

$$\lim_{j\to\infty} y_t(t-n_j, x) \underset{L^2(Q_p)}{=} \tilde{y}_t(t, x) \quad , \quad \lim_{j\to\infty} z_t(t-n_j, x) \underset{L^2(Q_p)}{=} \tilde{z}_t(t, x) . \tag{2.5}$$

Hence we may assume that it results, a.e. on $Q = J \times \Omega$,

$$\lim_{j\to\infty} y_t(t-n_j, x) = \tilde{y}_t(t, x) \quad , \quad \lim_{j\to\infty} z_t(t-n_j, x) = \tilde{z}_t(t, x) . \tag{2.6}$$

Let us observe that (since $\tilde{y}_{tt}(t, x)$, $\tilde{z}_{tt}(t, x) \in L^2(Q_p)$, $\forall p$) $\tilde{y}_t(t, x)$, $\tilde{z}_t(t, x)$ are continuous functions of $t \in J$, for all $x \in \Omega$, with the exception of those belonging to a set of measure zero. Furthermore, by (2.2), $\mathfrak{R}_{\tilde{y}(t)} \subseteq \overline{\mathfrak{R}}_{y(t)}$ and $\mathfrak{R}_{\tilde{z}(t)} \subseteq \overline{\mathfrak{R}}_{z(t)}$.

It is moreover, a.e. on Q,

$$a < \tilde{y}_t(t, x) < b \quad , \quad a < \tilde{z}_t(t, x) < b \tag{2.7}$$

and we may assume that it results, $\forall p$,

$$\begin{aligned} &\lim^*_{j\to\infty} \beta(y_t(t-n_j, x)) \underset{L^2(Q_p)}{=} \chi_1(t, x) \in \beta((\tilde{y}_t(t, x))^-) \,|\!\!-\!\!|\, \beta((\tilde{y}_t(t, x))^+), \\ &\lim^*_{j\to\infty} \beta(z_t(t-n_j, x)) \underset{L^2(Q_p)}{=} \chi_2(t, x) \in \beta((\tilde{z}_t(t, x))^-) \,|\!\!-\!\!|\, \beta((\tilde{z}_t(t, x))^+). \end{aligned} \tag{2.8}$$

The proof of (2.7), (2.8) is analogous to that given in [1], § 3, theorem III, and we shall omit it here.

We conclude that $\tilde{y}(t)$, $\tilde{z}(t)$ are solutions, on J, of the equation

$$\mathrm{A}u(t) - u''(t) + g(t) = \beta(u'(t)), \tag{2.9}$$

in the sense stated at § 1.

Let us prove now that $y(t) = z(t)$. Setting $w(t) = z(t) - y(t)$, we obtain the equation

$$\mathrm{A}w(t) - w''(t) = \beta(y'(t) + w'(t)) - \beta(y'(t)) \tag{2.10}$$

which implies, by the *energy* relation ($\forall t_1, t_2 \in \mathrm{J}$),

$$\| w(t_2) \|_{\mathrm{E}}^2 = \| w(t_1) \|_{\mathrm{E}}^2 - 2 \int_{t_1}^{t_2} (\beta(y'(t) + w'(t)) - \beta(y'(t)), w'(t))_{\mathrm{L}^2} \, \mathrm{d}t. \tag{2.11}$$

The function $\| w(t) \|_{\mathrm{E}}$, which is non negative and bounded, is therefore non increasing on J and it results

$$0 \leq \lim_{t \to +\infty} \| w(t) \|_{\mathrm{E}} = \mathrm{N}_1 \leq \lim_{t \to -\infty} \| w(t) \|_{\mathrm{E}} = \mathrm{N}_2 < +\infty, \tag{2.12}$$

where

$$\mathrm{N}_2^2 - \mathrm{N}_1^2 = 2 \int_{\mathrm{J}} (\beta(y'(t) + w'(t)) - \beta(y'(t)), w'(t))_{\mathrm{L}^2} \, \mathrm{d}t. \tag{2.13}$$

Consider now the sequence $\{w(t - n)\}$, $n = 0, 1, \cdots$

It results, by (2.2) and (2.3),

$$\begin{aligned} &\lim_{j \to \infty} w(t - n_j) \underset{\mathrm{E}}{=} \tilde{z}(t) - \tilde{y}(t) = \tilde{w}(t), \quad \text{uniformly on } \Delta_p, \forall p, \\ &\lim_{j \to \infty}{}^{*} w'(t - n_j) \underset{\mathrm{L}^2(\Delta_p; \mathrm{E})}{=} \tilde{w}'(t) \quad ; \quad \lim_{j \to \infty}{}^{*} \mathrm{A}w(t - n_j) \underset{\mathrm{L}^2(\Delta_p; \mathrm{L}^2)}{=} \mathrm{A}\tilde{w}(t) \end{aligned} \tag{2.14}$$

and moreover, a.e. on Q,

$$\lim_{j \to \infty} w_t(t - n_j, x) = \tilde{w}_t(t, x). \tag{2.15}$$

Let us prove, at first, that $\tilde{w}_t(t, x) = 0$ a.e. on Q. Assume, in fact, that $\tilde{w}_t(t, x) > 0$ on a set $\tilde{\mathrm{Q}}$, with $m(\tilde{\mathrm{Q}}) > 0$. We may, obviously, suppose that $\tilde{\mathrm{Q}}$ is closed and bounded and $\tilde{\mathrm{Q}} \subseteq (k \,|\!-\!|\, (k+1)) \times \Omega$, where k is a suitable integer: we assume, moreover, that all functions $y_t(t - n_j, x)$, $z_t(t - n_j, x)$ are continuous on $\tilde{\mathrm{Q}}$ and that the convergence, in (2.6), is uniform on $\tilde{\mathrm{Q}}$. It will therefore be

$$\tilde{w}_t(t, x) = \tilde{z}_t(t, x) - \tilde{y}_t(t, x) \geq 2\rho > 0 \qquad ((t, x) \in \tilde{\mathrm{Q}}) \tag{2.16}$$

and, consequently, when $j \geq \bar{j}$,

$$\begin{aligned} &z_t(t - n_j, x) - y_t(t - n_j, x) \geq \rho > 0, \\ &|z_t(t - n_j, x)| \leq \mathrm{M} \quad , \quad |y_t(t - n_j, x)| \leq \mathrm{M} \qquad (\mathrm{M} < +\infty). \end{aligned} \tag{2.17}$$

Let us consider now the function $\beta(\eta)$. It results $(\forall \xi, \eta \in a^{-\!-}b)$

$$\beta(\eta) - \beta(\xi) \geq \beta(\eta^-) - \beta(\xi^+) . \tag{2.18}$$

Hence, if $\eta \geq \xi + \rho$ (and assuming $\rho < b - a$),

$$\beta(\eta) - \beta(\xi) \geq \beta((\xi + \rho)^-) - \beta(\xi^+) > 0, \tag{2.19}$$

since $\beta(\eta)$ is strictly increasing. Let $|\xi|, |\eta| \leq M ; \eta - \xi \geq \rho$. It is

$$\beta((\xi + \rho)^-) - \beta(\xi^+) \geq \sigma > 0 . \tag{2.20}$$

In fact, if (2.20) did not hold, there would exist (also by (1.4)) a sequence $\{\xi_n\}$ such that $\lim\limits_{n\to\infty} \xi_n = \bar{\xi} \in a^{-\!-}b$, $\xi_n \neq \bar{\xi}$ and

$$\lim_{n\to\infty} \{\beta((\xi_n + \rho)^-) - \beta(\xi_n^+)\} = 0 . \tag{2.21}$$

However, if $\xi_n \downarrow \bar{\xi}$,

$$\lim_{n\to\infty} \{\beta((\xi_n + \rho)^-) - \beta(\xi_n^+)\} = \beta((\bar{\xi} + \rho)^+) - \beta(\bar{\xi}^+) > 0, \tag{2.22}$$

while, if $\xi_n \uparrow \bar{\xi}$,

$$\lim_{n\to\infty} \{\beta((\xi_n + \rho)^-) - \beta(\xi_n^+)\} = \beta((\bar{\xi} + \rho)^-) - \beta(\bar{\xi}^-) > 0. \tag{2.23}$$

Hence (2.20) holds. It follows, by (2.13) and (2.20),

$$\begin{aligned} N_2^2 - N_1^2 &= 2 \int_{-\infty}^{\infty} dt \int_{\Omega} (\beta(z_t(t,x)) - \beta(y_t(t,x)))(z_t(t,x) - y_t(t,x))\, dx \geq \\ &\geq 2 \sum_j^{\infty} \int_{\tilde{Q}} (\beta(z_t(t - n_j, x)) - \beta(y_t(t - n_j, x)))(z_t(t - n_j, x) - \\ &\qquad - y_t(t - n_j, x))\, dt\, dx \geq 2 \sum_j^{\infty} \rho\sigma m(\tilde{Q}) = \infty \end{aligned} \tag{2.24}$$

which is absurd. In the same way it may be shown that $\tilde{w}_t(t,x)$ cannot be < 0 on a set of positive measure.

Hence, a.e. on Q,

$$\tilde{w}_t(t,x) = 0, \tag{2.25}$$

that is $\tilde{w}(t,x)$ does not depend on $t : \tilde{w}(t,x) = \tilde{w}(x)$. Moreover, by (2.25), $w_{tt}(t,x) = 0$ a.e. on Q. Since $\tilde{w}(t,x)$ satisfies, by what has been proved above, the equation

$$A(x)\, w(t,x) - w_{tt}(t,x) = \chi_2(t,x) - \chi_1(t,x), \tag{2.26}$$

it results, consequently,

$$A(x)\, \tilde{w}(x) = \chi_2(t,x) - \chi_1(t,x) = h(x). \tag{2.27}$$

Let us prove that $h(x) = 0$, a.e. on Ω.

As already observed, $\tilde{y}_t(t, x), \tilde{z}_t(t, x)$ are continuous functions of $t, \forall x \in \Omega_0 \subseteq \Omega$, with $m(\Omega_0) = m(\Omega)$. Let us fix $\bar{x} \in \Omega_0$; if there exists $\bar{t} \in J$, such that $\beta(\eta)$ is continuous at $\bar{\eta} = \tilde{y}_t(\bar{t}, \bar{x}) = \tilde{z}_t(\bar{t}, \bar{x})$, then, by (2.8) and (2.27),

$$h(\bar{x}) = \chi_2(\bar{t}, \bar{x}) - \chi_1(\bar{t}, \bar{x}) = 0. \tag{2.28}$$

Otherwise, $\eta = \tilde{y}_t(t, \bar{x})$ will be a point at which β is not continuous, $\forall t \in J$. Assume now $h(\bar{x}) \neq 0$. There cannot exist two values t_1 and t_2 such that

$$\eta_1 = \tilde{y}_t(t_1, \bar{x}) < \tilde{y}_t(t_2, \bar{x}) = \eta_2, \tag{2.29}$$

$\beta(\eta)$ being discontinuous at η_1 and η_2. In fact, as $\tilde{y}_t(t, \bar{x})$ is continuous and β is strictly increasing, it would be $\beta(\eta_1^+) < \beta(\eta_2^-)$ and, if $\eta' \in \eta_1 \overline{\quad} \eta_2$ is a point at which $\beta(\eta)$ is continuous, there would exist $t' \in t_1 \overline{\quad} t_2$ such that $\tilde{y}_t(t', \bar{x}) = \eta'$, which would imply (against our assumption)

$$h(\bar{x}) = \chi_2(t', \bar{x}) - \chi_1(t', \bar{x}) = 0. \tag{2.30}$$

It results therefore, on all J, $\tilde{y}_t(t, \bar{x}) = \tilde{y}_t(t_1, \bar{x}) = \eta_1$, $\beta(\eta)$ being discontinuous at η_1. Hence, on all J,

$$\tilde{y}(t, \bar{x}) = \tilde{y}(t_1, \bar{x}) + (t - t_1)\, \tilde{y}_t(t_1, \bar{x}) \tag{2.31}$$

which implies necessarily (as $\tilde{y}(t)$ is L^2–bounded) $\tilde{y}_t(t_1, \bar{x}) = 0$, $\eta_1 = 0$; this is absurd since $\beta(\eta)$ is continuous at $\eta = 0$.

It follows $h(\bar{x}) = 0$ and, by (2.27), a.e. on Ω,

$$A\tilde{\tilde{w}}(x) = 0 \Rightarrow \tilde{\tilde{w}}(x) = 0. \tag{2.32}$$

Being

$$\|\tilde{\tilde{w}}\|_E = \lim_{j \to \infty} \|\tilde{w}(t - n_j)\|_E = N_2,$$

it follows $N_2 = 0$. Hence, by (2.13), $w(t) = z(t) - y(t) = 0$ on all J, and the theorem is proved.

3\. – II. (*Almost-periodicity theorem*). *Assume that* $f(t)$ *is* L^2–*w.a.p. in the sense* S^2 *of Stepanov and that* $\beta(\eta)$ *satisfies the hypotheses of theorem I. Assume, moreover, that there exists a bounded solution,* $y(t)$.

Then $y(t)$ *is* E–*a.p., while* $y'(t)$ *and* $Ay(t)$ *are respectively* E–S^2 *w.a.p. and* L^2–S^2 *w.a.p.*

The proof will be based on the classical procedure of Favard. Let $\{s_n\}$ be any real sequence. By (2.1), it is possible, as in theorem I, to select from $\{s_n\}$ a subsequence (which we shall again denote by $\{s_n\}$) such that it results, $\forall t \in J$,

$$\begin{aligned} &\lim_{n\to\infty} y(t + s_n) \underset{E}{=} z(t), \\ &\lim_{n\to\infty}{}^{*} y'(t + s_n) \underset{L^2(E)}{=} z'(t) \quad , \quad \lim_{n\to\infty}{}^{*} Ay(t + s_n) \underset{L^2(L^2)}{=} Az(t), \end{aligned} \tag{3.1}$$

where $L^2(L^2) = L^2(-1 \leq \eta \leq 0 ; L^2)$, $L^2(E) = L^2(-1 \leq \eta \leq 0 ; E)$; hence the third of (3.1), and analogously the second, means

$$(3.2) \qquad \lim_{n\to\infty} \int_{-1}^{0} (Ay(t+\eta+s_n), h(\eta))_{L^2} \, d\eta = \int_{-1}^{0} (Az(t+\eta), h(\eta))_{L^2} \, d\eta,$$

$\forall h \in L^2(L^2)$. We may assume moreover that it is, uniformly on J,

$$(3.3) \qquad \lim_{n\to\infty}{}^{*} f(t+s_n) \underset{L^2(L^2)}{=} g(t),$$

where $g(t)$ is L^2–S^2 w.a.p., as $f(t)$. As in [1], one proves that $z(t)$ is a bounded solution of the equation

$$(3.4) \qquad Az(t) - z''(t) + g(t) = \beta(z'(t)).$$

In order to prove, for instance, the E–almost–periodicity of $y(t)$, it will be sufficient, by Bochner's criterion, to show that the first of (3.1) holds uniformly on J. Assume, in fact, that this does not occur. There exist then $\rho > 0$ and three sequences $\{t_n\}$, $\{s_{n1}\} \subset \{s_n\}$, $\{s_{n2}\} \subset s_n$ such that it results

$$(3.5) \qquad \| y(t_n + s_{n2}) - y(t_n + s_{n1}) \|_E \geq \rho.$$

We may moreover assume that it is, uniformly on J,

$$(3.6) \qquad \lim_{n\to\infty}{}^{*} f(t+t_n+s_{n1}) \underset{L^2(L^2)}{=} \lim_{n\to\infty}{}^{*} f(t+t_n+s_{n2}) = q(t)$$

and, $\forall t \in J$,

$$(3.7) \qquad \begin{array}{lll} \lim\limits_{n\to\infty} y(t+t_n+s_{n1}) \underset{E}{=} z_1(t) & , & \lim\limits_{n\to\infty} y(t+t_n+s_{n2}) \underset{E}{=} z_2(t), \\ \lim\limits_{n\to\infty}{}^{*} y'(t+t_n+s_{n1}) \underset{L^2(E)}{=} z_1'(t) & , & \lim\limits_{n\to\infty}{}^{*} y'(t+t_n+s_{n2}) \underset{L^2(E)}{=} z_2'(t), \\ \lim\limits_{n\to\infty}{}^{*} Ay(t+t_n+s_{n1}) \underset{L^2(L^2)}{=} Az_1(t) & , & \lim\limits_{n\to\infty}{}^{*} Ay(t+t_n+s_{n2}) \underset{L^2(L^2)}{=} Az_2(t). \end{array}$$

The limit functions $z_1(t)$, $z_2(t)$ are then bounded solutions, on J, of the equation

$$Az(t) - z''(t) + q(t) = \beta(z'(t)).$$

By the uniqueness theorem of the bounded solution, it is therefore $z_1(t) = z_2(t)$, against (3.5). Hence the sequence $\{y(t+s_n)\}$ converges uniformly on J and $y(t)$ is E–a.p. In the same way the last part of the theorem can be proved.

References.

[1] L. Amerio and G. Prouse, *On the non linear wave equation with dissipative term discontinuous with respect to the velocity*, « Rend. Acc. Naz. dei Lincei », Note I e II (1968).