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A solution to Beniamino Segre's «Problem $\mathbf{I_{r,q}}$» for q even

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Geometria. — *A solution to Beniamino Segre's «Problem* $I_{r,q}$» *for q even.* Nota di LOUIS REYNOLDS ANTOINE CASSE, presentata (*) dal Socio B. SEGRE.

SUNTO. — Vengono stabiliti vari risultati concernenti i k-archi di $S_{r,q}$ con $r \geq 3$, q pari.

§ 1. INTRODUCTION.

We denote, as is usual, a finite linear space of dimension r over a Galois field GF (q) by $S_{r,q}$ [5].

DEFINITION 1. A *k_r-arc* (1) K_r of an $S_{r,q}$ is a set of k points of the $S_{r,q}$, no $r+1$ of which are linearly dependent, $k \geq r+1$. K_r is *complete* if it is not a proper subset of any $(k+1)_r$-arc of the $S_{r,q}$. A subspace $S_{s,q}$ of the $S_{r,q}$, $1 \leq s \leq r-1$, is *external*, *unisecant*, *bisecant*, $\cdots$, *u-secant* $\cdots$ to K_r depending on whether it has $0, 1, 2, \cdots, u, \cdots$ points in common with K_r. A bisecant line is also referred to as a *chord*. In the case of k_2-arcs, a unisecant line in the ambient $S_{2,q}$ is also called a *tangent* [1].

DEFINITION 2. A *tangent line* t at a point P of a k_3-arc K_3 of an $S_{3,q}$ is a unisecant line through P such that any plane containing t has at most one further point in common with K_3. A plane π containing a tangent line t at a point P of K_3, but containing no further point of K_3, is called an *osculating plane* of K_3 at P.

DEFINITION 3. A k_2-arc K_2 of an $S_{2,q}$ for which k attains its maximum value for the $S_{2,q}$ is called an *oval* [1]. A k_3-arc K_3 of an $S_{3,q}$ for which k attains its maximum value for the given $S_{3,q}$ is called a *cubal*.

Numerous papers have appeared on the subject of k_2-arcs, mainly by B. Segre and his school. One problem proposed by B. Segre was [1]:

PROBLEM $I_{r,q}$: For given r and q, what is the maximum value, denoted by $|k|$, of k for which k_r-arcs exist in $S_{r,q}$. And what, precisely, are the k_r-arcs corresponding to such a value of k?

(*) Nella seduta del 19 novembre 1968.

(1) The term 'k-arc' is used instead of 'k_r-arc' in the various papers to which reference is made.

The results contained in this paper are from the author's doctoral thesis approved by the University of London for the award of the Ph. D. degree. In this connection, the author wishes to thank his supervisor, Dr. E. Stein, for her encouragement and valuable advice.

He produced the following answers [5]

(1) If q is odd, every $(q+1)_2$–arc is a conic.

(2) If q is odd, every $(q+1)_3$–arc is a twisted cubic.

(3) If q is odd, $r = 2, 3$ or 4, then $|k| = q+1$.

(4) If q is odd, $r > 4, q \geq r+2$ then $|k| \leq q+r-3$.

(5) If q is even, $r = 2$, then $|k| = q+2$.

(6) If q is even, $r > 2$, then $|k| \leq q+r$.

In the present paper, we shall prove:

(1) If q is even, $r = 3$ or 4, $q \geq r+1$, then $|k| = q+1$.

(2) The tangent lines to a $(q+1)_3$–arc of an $S_{3,q}$, $q = 2^h$ are the generators of an hyperbolic quadric. If $q=4$ or 8, a $(q+1)_3$–arc is a twisted cubic.

We now quote three theorems which we shall require later.

THEOREM 1.1. *If $q = 2, 4$ or 8, every oval of an $S_{2,q}$ is made up of $q+1$ points of a conic C and of the nucleus N of C* [4].

THEOREM 1.2. *If q is even, any two $(q+2)_2$–arcs of an $S_{2,q}$ coincide if they have more than half their number of points in common* [10].

THEOREM 1.3. *If q is even, and if $k > q - \sqrt{q} + 1$, then every k_2–arc is contained in a $(q+2)_2$–arc; this $(q+2)_2$–arc is unique but for one exception $q = 2, k = 2$* [7].

§ 2. k_3–ARCS OF $S_{3,q}$.

It follows from definitions 1 and 2 that through a point Q not belonging to a k_3–arc K_3 there cannot pass

(*a*) two chords.

(*b*) a tangent line and a chord.

THEOREM 2.1. *If Q is a point not belonging to a k_3–arc K_3, then the necessary and sufficient condition for $K_3 + Q$ to be a $(k+1)_3$–arc is that the lines joining Q to the points of K_3 are tangent lines to K_3.*

Proof: Let P_i $(i = 1, 2, \cdots, k)$ be the points of K_3. Suppose that QP_i is a tangent line at P_i, for all i. Then, by definition 2, Q does not belong to any of the $\binom{k}{3}$ planes spanned by the points of K_3. Thus $K_3 + Q$ is a $(k+1)_3$–arc.

Suppose next that $K_3 + Q$ is a $(k+1)_3$–arc. If, for some i, QP_i is not a tangent line to K_3, then there exists at least one plane containing QP_i and two points (other than P_i) of K_3; in which case, $K_3 + Q$ is not a $(k+1)_3$–arc. Hence QP_i is a tangent line to K_3.

THEOREM 2.2. *A k_3-arc K_3 has at most $q + 3$ points.*

Proof: Each of the $q + 1$ planes containing a chord of K_3 contains at most one further point of K_3.

Project a k_3-arc K_3 from a point P of itself onto a plane π (not through P) and consider the $(k - 1)_2$-arc K_2 so obtained. Suppose that K_3 has a tangent line l at P, and let l intersect π in O. K_2 is incomplete since O can be added to it so as to form a k_2-arc C of π. At O, C has exactly $q - k + 2$ tangents in π. If t is a tangent to C at O, then the plane $l \cdot t$ has only the point P in common with K_3. Therefore the $q - k + 2$ planes of type $l \cdot t$ are the osculating planes of K_3 at P containing the tangent line l. Let O_i be any point in π not belonging to the projection K_2. If $K_2 + O_i$ is a k_2-arc, then $O_i P$ is a tangent line of K_3 at P. Thus the number of such points O_i in π is equal to the number of tangent lines of K_3 at P.

In particular, if K_2 is complete, there are no tangent lines of K_3 at P, and it then follows by theorem 2.1 that K_3 is complete.

Nothing so far suggests that if a k_3-arc K_3 has a certain number of tangent lines at one of its points P, then K_3 needs to have the same number (or any) tangent lines at some or all of its remaining points.

THEOREM 2.3. *If q is even and if $k > q - \sqrt{q} + 2$ (with the only exception $q = 2$, $k = 3$), then*

(1) *at every point of a k_3-arc K_3 of $S_{3,q}$ there pass the same number n of tangent lines, where*

$$n = q - k + 3$$

(2) *at every point of K_3 there pass the same number m of osculating planes, where*

$$m = (q - k + 2)\, n - \binom{n}{2} \qquad \textit{if } n > 1$$

or

$$m = (q - k + 2)\, n \qquad \textit{if } n \leq 1.$$

Proof: Project K_3 from any point P of itself onto some plane π not through P. Under the condition $k > q - \sqrt{q} + 2$, every $(k - 1)_2$-arc is incomplete and can be uniquely completed to an oval (see Theorem 1.3). This holds for any point P of K_3 taken as vertex of projection. Thus, at each point of K_3 there pass

$$q + 2 - (k - 1) = q - k + 3$$

tangent lines.

As was noted earlier, there are $q - k + 2$ osculating planes containing each tangent line of a k_3-arc K_3. Also, every pair of tangent lines of K_3 at one of its points P forms an osculating plane. Hence the number m of osculating planes is

$$m = (q - k + 2)\, n - \binom{n}{2} \qquad \text{if } n \geq 2$$

or

$$m = (q - k + 2)\, n \qquad \text{if } n \leq 1.$$

COROLLARY 2.1. *A $(q+3)_3$–arc possesses no tangent line; a $(q+2)_3$–arc possesses no osculating planes but has exactly one tangent line at each of its points; a $(q+1)_3$–arc has exactly two tangent lines at each of its points, and the plane containing the pair of tangent lines at each of its points is the osculating plane there.*

THEOREM 2.4. *A $(q+3)_3$–arc* K_3 *has no bisecant planes.*

Proof: About any chord of a $(q+3)_3$–arc K_3 there pass exactly $q+1$ planes: these planes are necessarily trisecant planes of K_3.

THEOREM 2.5. *Given a $(q+3)_3$–arc* K_3 *of* $S_{3,q}$, $q>2$, *there exist points not belonging to any chord of* K_3.

Proof: The number N of points not belonging to any chord of a $(q+3)_3$–arc is

$$\begin{aligned} N &= q^3+q^2+q+1-\binom{q+3}{2}\times(q-1)-(q+3) \\ &= (q+1)(q-1)(q-2)/2 \end{aligned}$$

Thus N is positive if $q>2$.

§ 3. CUBALS AND THE PROBLEM $I_{r,q}$.

LEMMA 3.1. *If q is even, a $(q+2)_3$–arc (if it exists) is incomplete and can be uniquely completed to a $(q+3)_3$–arc.*

Proof: Single out any point of a $(q+2)_3$–arc K_3 and call it P. Call the remaining points of K_3, P_i $(i=1,2,\cdots,q+1)$, and denote by K_3' the $(q+1)_3$–arc formed by the points P_i. Now K_3 has exactly one tangent line at each of its points (see corollary 2.1): denote by l, l_i respectively the tangent lines of K_3 at P, P_i. By theorem 2.1 the line PP_i is a tangent line at P_i to K_3'. Thus the osculating plane of K_3' at P_i is the plane determined by PP_i and l_i. But the osculating plane of K_3' at P_i is also the plane determined by the lines PP_i and l (since this plane contains the tangent line PP_i of K_3' and contains no further point of K_3'). Therefore l intersects l_i.

In other words, any two tangent lines of a $(q+2)_3$–arc intersect in a point. Thus the tangent lines of a $(q+2)_3$–arc are either coplanar or they are concurrent in a point Q, say. The first possibility is contradictory to the definition of tangent lines. By theorem 2.1, Q is the unique point which can be added to K_3 to form a $(q+3)_3$–arc.

THEOREM 3.1. *If q is even, $q\neq 2$, there do not exist $(q+3)_3$–arcs.*

Proof: Let K be a $(k;3)$–arc (2) of an $S_{3,q}$. Let

s = the number of trisecants of K.

n_i = the number of trisecants of K through a point P_i of K.

(2) A $(k;3)$–arc K is a set of k coplanar points, no four of which are collinear, but K contains at least one set of three collinear points; a trisecant of K is a line which has three points in common with K [8].

As proved in [8]

$$(3.1) \qquad \sum_{i=1}^{k} n_i = 3\,s$$

Suppose there exists a $(q+3)_3$–arc K_3. Let A_i $(i = 0, 1, 2, \cdots, q+2)$ be the points of K_3. Two cases are to be considered.

Case 1: $q = 2^{2h}$. Project K_3 from any point O of $A_0 A_1$ (distinct from A_0 and A_1) onto some plane π not through O. Let $O A_0 A_1$ and OA_i $(i = 2, 3, \cdots, \cdots, q+2)$ intersect π in the points P_1 and P_i respectively. In π, the points P_i $(i = 1, 2, \cdots, q+2)$ form a $(q+2\,;3)$–arc. Since each of the $q+1$ planes containing the line $OA_0 A_1$ passes through one point of K_3, it follows (in the above notation) that $n_1 = 0$. Consider the planes containing a line OA_i. Since none of these planes are bisecant of K_3 (theorem 2.4), it follows that the points of K_3 distinct from A_0, A_1 and A_i divide themselves into $q/2$ pairs, each pair of points being coplanar with A_i and O.

Hence, for all i, we have $n_i = q/2$. Applying equation (3.1), we have:

$$3\,s = 0 + (q+1)\left(\frac{q}{2}\right).$$

Hence $q + 1 \equiv 0$ (mod. 3). But $2^{2h} + 1$ is not divisible by $2 + 1$.

Therefore if $q = 2^{2h}$, there do not exist $(q+3)_3$–arcs.

Case 2: $q = 2^{2h+1}$, $h \neq 0$. Choose a point V not lying on any chord of K_3 (by Theorem 2.5, if $q \neq 2$, there exists such a point). Project K_3 from V onto some planer π not through V. Let P_i be the projection of A_i $(i = 0, 1, \cdots, q+2)$. Since K_3 has no bisecant planes, it follows that the $(q+3\,;3)$–arc just obtained is such that $n_i = \frac{1}{2}(q+2)$. Again, applying equation (3.1), we have:

$$3\,s = (q+3) \times \frac{1}{2}(q+2)$$

Hence $\quad q^2 + 5q + 6 \equiv 0 \pmod 6$

i.e. $\quad q + 5 \equiv 0 \pmod 3$

i.e. $\quad q - 1 \equiv 0 \pmod 3$

But $2^{2h+1} - 1$ is not divisible by $2 + 1$. Therefore if $q = 2^{2h+1}$ there do not exist $(q+3)_3$–arcs.

The proof is now complete.

Any set of five points, no four of which are coplanar, can be transformed by a unique homography to the five points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$ and $(1, 1, 1, 1)$. Therefore if $q = 2$, a cubal is a $(q+3)_3$–arc, and any two cubals are projectively equivalent.

THEOREM 3.2. *If* $q \neq 2$, *a cubal is a* $(q+1)_3$–*arc*.

Proof: This follows from lemma 3.1 and theorem 3.1.

THEOREM 3.3. *If q is even, $q \neq 2$, the $2(q+1)$ tangent lines of a cubal are the $2(q+1)$ generators of a hyperbolic quadric.*

Proof: Let $A_1, A_2, \cdots, A_{q+1}$ be the points of a cubal K_3 of $S_{3,q}$, q even, $q \neq 2$. Let l_i, m_i be the two tangent lines of K_3 at the point A_i ($i = 1, 2, \cdots \cdots, q+1$). Consider the planes containing the chord $A_i A_j$, for some i, j. Of these, $q-1$ are trisecant planes and the remaining two are bisecant planes. The two bisecant planes can be identified either as $(A_i A_j) \cdot m_j$ and $(A_i A_j) \cdot l_j$ or as $(A_i A_j) \cdot m_i$ and $(A_i A_j) \cdot l_i$. Hence, one tangent line at A_i intersects one tangent line at A_j, and the second tangent line at A_i intersects the second tangent line at A_j (by definition, the osculating plane at A_i does not pass through A_j, and therefore a tangent line at A_j does not intersect both tangent lines at A_i).

By renaming the tangent lines if necessary, let the tangent lines of K_3 which intersect l_1, m_1 be m_j, l_j respectively ($j = 2, 3, \cdots, q+1$). We now show that the $q+1$ lines m_i ($i = 1, 2, \cdots, q+1$) are skew to each other. For suppose m_j and m_k intersect, $j \neq k \neq 1$. Two possibilities may occur: either l_1, m_j, m_k are coplanar or they are concurrent in a point Q distinct from A_1 (since both m_j, m_k intersect l_1). The first possibility contradicts the definition of a tangent line. So, suppose that l_1, m_j, and m_k are concurrent in a point Q. Consider any point A_i of K_3, ($i \neq 1, j, k$). Again since no three tangent lines of K_3 are coplanar, and since m_i intersects l_1 it follows that m_i intersects m_j or m_k if and only if m_i passes through Q. If m_i does not intersect m_j, m_k then l_i does. In that case l_i passes through Q. This is impossible since l_i, m_i do not both intersect l_1. Thus the assumption that l_1, m_j, m_k are concurrent in a point Q leads to the conclusion that all the tangent lines m_i ($i \neq 1, i = 2, \cdots, q+1$) pass through Q. By theorem 2.1, $K_3 + Q$ is a $(q+2)_3$-arc. This is impossible: by theorem 3.1 and lemma 3.1, there do not exist $(q+2)_3$-arcs.

It follows that each member of the set of tangent lines l_i meets every member of the set of tangent lines m_i, and that no two members of the same set intersect. Consider any three members of the set $l_i : l_1, l_2, l_3$ say. The (unique) hyperbolic quadric Q_2^2 containing the lines l_1, l_2, l_3 contains the whole set of tangent lines m_i (since the points $m_i \cdot l_j$ belong to Q_2^2, $j = 1, 2, 3$), and therefore contains all the tangent lines l_i.

THEOREM 3.4. *Every cubal K_3 of $S_{3,q}$, $q = 4$ or 8, is a twisted cubic.*

Proof: Any set of five points of $S_{3,4}$, no four of which are coplanar, can be transformed by a unique homography into $(1, 0, 0, 0)$, $(0, 0, 0, 1)$, $(1, 1, 1, 1)$, $(1, \omega, \omega^2, 1)$ and $(1, \omega^2, \omega, 1)$, where ω is a generator of GF(4). Thus every cubal of $S_{3,4}$ is projectively equivalent to the twisted cubic $(\theta^3, \theta^2, \theta, 1)$.

Suppose next that $q = 8$. Let $P_1, \cdots, P_9$ be the nine points of a cubal K_3 of $S_{3,8}$. Denote by Q_2 the hyperbolic quadric whose generators are the tangent lines of K_3 (theorem 3.3). Project from P_1 onto some plane π not through P_1. Denote the projection of P_i by P'_i ($i = 2, 3, \cdots, q+1$). Let g_1, g_2 be the two generators of Q_2 through P_1; let g_1, g_2 intersect π in A_1, A_2 respectively. Thus $A_1, A_2, P'_2, P'_3, \cdots, P'_9$ form a 10_2–arc of π. By theorem 1.1, this oval is made up of 9 points of a conic C and of the nucleus N of C. Denote by $P_1(C)$ the quadric cone projecting C from P_1. If neither A_1 nor A_2 coincides with the nucleus N of C, then $P_1(C)$ and Q_2 intersect in g_1, g_2 and residually in a conic; this is impossible since the points of K_3 are not coplanar. So, suppose without loss of generality that A_2 is the nucleus of C. Hence $P_1(C)$ and Q_2 intersect in g_1 and residually in a cubic curve C^3 containing K_3. Since K_3 has more than five points, C^3 is neither a triad of lines nor a conic plus a line, and therefore C^3 is a twisted cubic. The theorem now follows since a twisted cubic in $S_{3,8}$ has exactly 9 points (i.e. as many points as K_3).

THEOREM 3.5. *Denoting by* $|k|$ *the maximum value for which there exist* k_r*–arcs in* $S_{r,q}$, $q = 2^h$, $q \geq r + 1$, $r \geq 4$ *then*

(1) $$|k| = q + 1 \qquad \text{if } r = 4$$

(2) $$|k| \leq q + r - 3 \qquad \text{if } r > 4$$

Proof: Consider a k_4–arc K_4 of $S_{4,q}$, $q = 2^h$, $q \geq 8$. Project K_4 from an arbitrarily chosen point P of K_4 onto some $S_{3,q}$ not through P. The projection is a $(k - 1)_3$–arc of the $S_{3,q}$, and by theorem 3.2, we have $|k| \leq q + 2$.

Suppose there exists a $(q + 2)_4$–arc K_4 of $S_{4,q}$, $q = 2^h$, $q \geq 8$. Let the points of K_4 be $O_1, O_2, P_1, P_2, \cdots, P_q$. Project successively K_4 from O_1 and O_2 onto some $S_{3,q}$ not through O_1 or O_2. The projections are cubals $K_3^{(i)}$ $(i = 1, 2)$ of the $S_{3,q}$, and by theorem 3.3 the tangent lines of the cubal $K_3^{(i)}$ are the generators of a hyperbolic quadric which we denote by Q_i. Denote by $O_i(Q_i)$ the quadric of the ambient $S_{4,q}$ having O_i as vertex and Q_i as base $(i = 1, 2)$. Let $O_1 O_2$ intersect the $S_{3,q}$ in the point O, and let $O_i P_j$ intersect the $S_{3,q}$ in $P_j^{(i)}$ $(i = 1, 2, \cdots, q)$. If follows that for all j, O, $P_j^{(1)}$, $P_j^{(2)}$ are collinear; denote the line joining them by l_j. Lastly, denote the two tangent lines of $K_3^{(1)}$ at the point O by l and m. Now the q planes formed by l and the lines l_j are bisecant planes not only of $K_3^{(1)}$ but also of $K_3^{(2)}$. Thus l is a tangent line of $K_3^{(i)}$ at O. Similarly m is a tangent line of $K_3^{(2)}$ at O. Thus the quadrics $O_1(Q_1)$ and $O_2(Q_2)$ intersect in the plane formed by l and $O_1 O_2$ and in the plane formed by m and $O_1 O_2$; therefore they intersect residually in an algebraic variety V_2^2 of order two and dimension two. This residual intersection V_2^2 is thus properly contained in a three-dimensional linear space; but V_2^2 also contains at least q points of K_4 (namely $P_1, P_2, \cdots, P_q$). This is a contradiction: by definition, not more than four points of K_4 can be contained in a three-dimensional linear space. Hence there do not exist $(q + 2)_4$–arcs in $S_{4,q}$. Now the quartic curve $(\theta^4, \theta^3, \theta^2, \theta, 1)$ has exactly $q + 1$ points. Therefore

$$|k| = q + 1 \qquad \text{if } r = 4.$$

Lastly, consider a k_r–arc K_r of an $S_{r,q}$, $q = 2^h$, $q \geq r + 1$, $r > 4$. Choose arbitrarily $r - 4$ points of K_r and project K_r from the $S_{r-5,q}$ spanned by these points onto some $S_{4,q}$ skew to the $S_{r-5,q}$. The projection is a $(k - r + 4)_4$–arc, which has at most $q + 1$ points. Hence

$$|k| \leq q + r - 3 \quad \text{if} \quad r > 4.$$

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