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Gauge transformations and conservationLaw s (*)

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Fisica matematica. — *Gauge transformations and conservation Laws* (*). Nota di ENZO TONTI, presentata (**) dal Socio B. FINZI.

RIASSUNTO. — La relazione esistente tra la invarianza di gauge e le leggi di conservazione è usualmente dimostrata per mezzo del teorema di Noether. In questa nota dimostro che questa relazione può essere dedotta dalla struttura matematica delle teorie lineari di campo per mezzo delle condizioni di esistenza e di unicità delle soluzioni delle equazioni di campo.

1. MATHEMATICAL PRELIMINARIES. - By *differential operator* we mean a *formal differential operator* plus a *domain*. A *formal differential operator* is an expression containing derivation symbols and known functions and it is obtained from a differential equation by striking out the unknown functions, i.e. the operand, on which the formal operator is applied. When we prescribe the set of functions on which we intend to apply the formal operator, specifying the boundary and initial conditions, the derivability requirements, the field of definition of the independent variables and so on, we have chosen a *domain* (1). We shall denote an operator with a capital block letter, say L, its domain with D(L) and the corresponding formal operator with an italic capital letter, say $\mathfrak{L}$. An operator is called *linear* if its formal operator is linear and the domain is a linear set [1; *24*]. Functions that satisfy homogeneous boundary or initial conditions form a linear set, but if they satisfy prescribed non homogeneous conditions they do not form a linear set: happily it is always possible to transform non homogeneous conditions into homogeneous ones by changing the function on which the operator works.

The *formal adjoint* of a linear formal differential operator $\mathfrak{L}$ in the linear formal operator $\tilde{\mathfrak{L}}$ defined, in the real case, by the relation

$$\int_\Omega v\mathfrak{L}u \, \mathrm{d}\Omega = \int_\Omega u\tilde{\mathfrak{L}}v \, \mathrm{d}\Omega + \{\text{boundary terms}\} \tag{1}$$

where Ω is the field of definition of the independent variables. The *adjoint* of a linear differential operator L is the operator $\tilde{\mathrm{L}}$ having as a formal part $\tilde{\mathfrak{L}}$ and as a domain the set of functions that satisfy the minimum number of boundary or initial conditions necessary to make the boundary terms in Eq. (1) vanish. Then the following relation holds

$$\int_\Omega v\mathrm{L}u \, \mathrm{d}\Omega = \int_\Omega u\tilde{\mathrm{L}}v \, \mathrm{d}\Omega . \tag{2}$$

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(1) This distinction between formal operator and operator is used by some authors as Stone [2], Dunford–Schwartz [3] but, unhappily is not commonly used in the applications of mathematics to physics.

If u' and u'' are two functions or vectors or tensors of an Hilbert (function or vector or tensor) space the scalar product is defined as

$$(u', u'') \stackrel{\text{def}}{=} \int_{\Omega} u'(x)\, u''(x)\, d\Omega \tag{3}$$

where x denotes concisely the independent variables, and $u'(x)\, u''(x)$ is the inner product of the two tensors.

A differential operator can operate only on those functions that have a minimum of derivability requirements: therefore its domain cannot be the whole of an Hilbert tensor space. Besides, if we desire that the transformed functions $v = Lu$ belong to another Hilbert space, they must have a finite norm. But because differential operators may be unbounded we must leave out of the domain those tensors u which are transformed into tensors v of infinite norm. A sufficient condition in order that the generated tensors v have a finite norm is that the corresponding tensor u possess continuous derivatives in the field Ω and on its surface up to the highest order of the derivatives appearing in the differential operator [1, *51*]. An equation plus additional conditions will be called a *problem*. If the formal operator is linear and if the additional conditions are homogeneous then the entire operator is linear and we have a linear problem. Given the linear problem

$$Lu = f \tag{4}$$

we ask ourselves: what conditions must be satisfied by f in order to assure the existence of the solution? [(2)]. The answer lies in the following general theorem of functional analysis [4, *172*]: *Existence theorem*: the *necessary* condition in order that the linear problem $Lu = f$ have a solution is that f be orthogonal to *all* solutions of the adjoint homogeneous problem $\tilde{L}v = 0$ i. e. if v_0 is an arbitrary solution of this problem

$$(f, v_0) = 0. \tag{5}$$

If the range is closed the condition becomes sufficient. To investigate the closure of the range is usually the difficult point for the existence proof. For our purposes the sufficiency is not of interest and then we disregard it in the following.

In many cases we know particular or general solutions $v_0(x)$ of the adjoint homogeneous problem, and then condition (5) can be rendered explicit. When we know a *particular solution*, condition (5) expresses a global conservation law not reducible to a local one because the field Ω is the entire field of variability of the independent variable and not an arbitrary field $\omega \subset \Omega$. When we know instead a *general solution*, condition (5) can be transformed into a differential condition, thus expressing a *local* compatibility condition.

(2) Some precisation about terminology: *conditions of existence* of the solution of a problem embodies *compatibility conditions* for source term: these express, from the physical point of view, *conservation laws*.

We must define what we mean by *general* solution of an equation and of a problem. By general solution of a homogeneous system of equations $\mathfrak{L}u = 0$ we mean an expression of the kind $u = \mathfrak{R}\chi$ where χ indicates an arbitrary tensor of given rank, called the *potential*, $\mathfrak{R}$ indicates a formal differential operator, and such that the following two requirements are satisfied

1) if we insert the tensor u given by $u = \mathfrak{R}\chi$ into the system $\mathfrak{L}u = 0$ this is identically satisfied: $\mathfrak{L}\mathfrak{R}\chi \equiv 0$. Then $u = \mathfrak{R}\chi$ is a *solution*.

2) *all* solutions of the system $\mathfrak{L}u = 0$ are contained in $u = \mathfrak{R}\chi$ and can be obtained by specializing the potential χ: then the solution is *general*. This requires the proof of the " completeness " of the system of functions under some prescribed derivability requirements of the function u in $\mathrm{L}u = 0$.

It is immediate to show that from the fact that $\mathfrak{L}\mathfrak{R} \equiv 0$ it follows $\tilde{\mathfrak{R}}\tilde{\mathfrak{L}} \equiv 0$. In fact if u and v are two tensors vanishing on the boundary with their derivatives we have

$$0 \equiv (v , \mathfrak{L}\mathfrak{R}u) = (\tilde{\mathfrak{L}}v , \mathfrak{R}u) = (\tilde{\mathfrak{R}}\tilde{\mathfrak{L}}v , u)$$

for every u and v: then $\tilde{\mathfrak{R}}\tilde{\mathfrak{L}} \equiv 0$. For example from the fact that div(curl) $\equiv 0$ it follows curl (— grad) $\equiv 0$ as is well known.

By general solution of a *problem* $\mathrm{L}u = 0$ we mean the general solution of the corresponding system $\mathfrak{L}u = 0$ with additional conditions on the potential χ which guarantee that $u = \mathfrak{R}\chi$ belongs to the domain of the operator L. With such condition on χ we obtain an operator R: then the expression $u = \mathrm{R}\chi$ is called the *general solution* of the *problem* $\mathrm{L}u = 0$.

With this premise we show that when a general solution of the adjoint homogeneous equation $\tilde{\mathfrak{L}}v = 0$ is known, the orthogonality condition (5) gives rise to a local compatibility condition for f.

In fact if $v_0 = \mathfrak{S}\varphi$, it will be

$$0 = \int_\Omega f v_0 \, d\Omega = \int_\Omega f \mathfrak{S}\varphi \, d\Omega = \int_\Omega \varphi \tilde{\mathfrak{S}} f \, d\Omega + \oint_S \alpha\varphi \;\; \beta f \, dS = 0 \tag{6}$$

being α and β two formal operators.

For the fundamental lemma of the calculus of variations must be

$$\tilde{\mathfrak{S}} f = 0 \, ; \tag{7}$$

this is a local compatibility condition. Moreover, if we know the general solution of the *problem* $\tilde{\mathrm{L}}v = 0$ we obtain again Eq. (7) as necessary condition and because boundary term in Eq. (6) must vanish on account of (7), we have two cases: either the boundary conditions on the potential φ are such that $\alpha\varphi = 0$ on S and in this case boundary terms vanish automatically or $\alpha\varphi$ does not vanish on S or on a part of it. In the latter case

$$\beta f = 0 \qquad \text{must hold on S} \tag{8}$$

or on a part of it. This condition is necessary as is Eq. (7) and it is a local *boundary* compatibility condition for problem (4).

Concerning the uniqueness of solutions of the problem (4) we have the following theorem [4, *166*].

Uniqueness theorem. The necessary and sufficient condition for the solution of the problem $Lu = f$ *to be unique is that the homogeneous problem* $Lu = 0$ *have only the trivial solution* $u = 0$.

2. STRUCTURE OF LINEAR FIELD THEORIES. – Many linear field theories exhibit the following mathematical structure: when the field equations are expressed by means of two systems of differential equations, say

(9 *a*) $\mathcal{A}u = f$ (9 *b*) $\mathcal{B}v = g$

the general solution of one set has as formal operator the formal adjoint of the formal operator that forms the other system. This means that the general solutions of Eq. (9 *a*) and (9 *b*) are respectively

(10 *a*) $u = \tilde{\mathcal{B}}\varphi + u_f$ (10 *b*) $v = \tilde{\mathcal{A}}\psi + v_g$

where u_f and v_g are two particular solutions of the non homogeneous equations.

Moreover in time independent fields not only the formal operators, but also the operators are mutually adjoint i.e. given the two problems

(11 *a*) $Au = f$ (11 *b*) $Bv = g$

their general solutions are:

(12 *a*) $u = \tilde{B}\psi + u_f$ (12 *b*) $v = \tilde{A}\varphi + v_g$

This structure is at the origin of many mathematical properties of the theory and as consequence of many physical properties. For example this structure is at the origin of the variational principles of the field theory [5], [6] of the reciprocity theorems and generalized forms of the principle of virtual work [7] and of the relation between *gauge invariance and conservation laws*, as we show in this paper.

We start from the second case where the operators, and not only the formal operators, are tied together as in Eq. (11) and Eq. (12).

In order that the problem (11 *a*) admit a solution, f must be orthogonal to all the solutions of the adjoint homogeneous problem. But at this point the structure mentioned before becomes operative: the adjoint homogeneous problem is formed with the operator $\tilde{A}$ appearing in the general solution of the associated problem (12 *b*). If this homogeneous problem admits a general solution, say $\varphi_0 = R\chi$, then the associated problem (12 *b*) has many solutions as a consequence of the uniqueness theorem. Thus as a consequence of the adjointness of the operators the *existence of the solution for one set is related to the uniqueness solution of the associated set and viceversa.*

Moreover the fact that the associated problem (12 *b*) has many solutions implies that it is not changed by the transformation $\varphi \to \varphi + R\chi$: this invariance is known in theoretical physics as *gauge invariance* of the field equations. Since the compatibility conditions on f mean conservation laws for the source term, we can conclude that as a consequence of the structure *the existence of the gauge invariance for one set embodies conservation law for the source term of the remaining set and viceversa.* Furthermore being $\tilde{A}R \equiv 0$ it will be $\tilde{R}A \equiv 0$ and then $\tilde{R}f = 0$. Then *the operator that forms the local compatibility conditions for the source term of one problem is just the adjoint of the operator that gives the gauge invariance of the associated problem.*

When the adjoint homogeneous problem has only particular solutions, the operator R is not differential and then we have only global conservation for f.

All we have said can be equally applied to the existence condition of the problem (11 *b*) which will be related with the uniqueness of the associated problem (12 *a*). When the structure of a linear field theory we have spoken of is restricted to the formal operator all we have said holds true with the only substitution of the operator R with the formal operator $\mathfrak{R}$.

The equations

$$\tilde{\mathfrak{R}}\mathfrak{A} \equiv 0 \tag{13}$$

are the so called Bianchi type identities for the system $\mathfrak{A}u = f$.

3. VECTOR FIELD. – We consider a vector field described by two vectors $\vec{u}$ and $\vec{v}$ related with the linear relation $\vec{v} = a\vec{u}$ and satisfying the problems

$$(14\,a)\quad \begin{cases} \operatorname{div} \vec{u} = f \\ \vec{n}\cdot\vec{u} = 0 \quad \text{on } S_u \end{cases} \qquad (14\,b)\quad \begin{cases} \operatorname{curl} \vec{v} = \vec{g} \\ \vec{n}\times\vec{v} = 0 \quad \text{on } S_v \end{cases}$$

where S_u and S_v denotes two pieces of the surface S one of which can coincide with the entire surface. The general solutions (3) of the corresponding homogeneous problems are

$$(15\,a)\quad \begin{cases} \vec{u} = \operatorname{cur} \vec{\psi} \\ \vec{n}\times\vec{\psi} = 0 \quad \text{on } S_u \end{cases} \qquad (15\,b)\quad \begin{cases} \vec{v} = -\operatorname{grad} \varphi \\ \varphi = 0 \quad \text{on } S_v . \end{cases}$$

It is easy to show that the general solution of one problem has as operator the adjoint of the operator of the associated problem. In fact from the relation:

$$\int_\Omega \varphi\,[\operatorname{div}]\,\vec{u}\,d\Omega = \int_\Omega \vec{u}\cdot[-\operatorname{grad}]\,\varphi d\Omega + \oint_S \varphi\vec{n}\cdot\vec{u}dS \tag{16}$$

being $\vec{n}\cdot\vec{u} = 0$ on S_u it must be $\varphi\vec{n}\cdot\vec{u} = 0$ on S_v. But this implies $\vec{n}\varphi = \vec{\lambda}\times\vec{u}$ or equivalently $\varphi = \vec{\lambda}\cdot(\vec{n}\times\vec{u})$, with $\vec{\lambda}$ arbitrary. Because $\vec{v} = a\vec{u}$ we have

(3) That the solutions are *general* under specified derivability requirements of the arguments was shown by Lichtenstein [10, *101*].

$\varphi = \vec{\lambda}/a \cdot (\vec{n} \times \vec{v})$. Now $\vec{n} \times \vec{v} = 0$ on S_v and then $\varphi = 0$. Then the two operators that form the problems (14 *a*) and (15 *b*) are mutually adjoint.
In an analogous way we can show that also the operators of (14 *b*) and (15 *a*) are mutually adjoint. Then the structure of Sec. 2 is here realized.

We ask now ourselves what are the compatibility conditions for the source term f. To this end we must find the solution of the adjoint homogeneous problem

$$(17) \qquad \begin{cases} -\operatorname{grad} \varphi = 0 \\ \varphi = 0 \quad \text{on } S_u . \end{cases}$$

If $S_u \neq 0$ the only solution is $\varphi_0 = 0$ and then the existence condition (5) is satisfied by every f and we have no compatibility conditions for f. If $S_u = 0$, we have the solution $\varphi = C$ and condition (5) becomes: $\int_{\Omega} f \, d\Omega = 0$. This is a global conservation law.

We now ask ourselves the compatibility conditions for $\vec{g}$. The adjoint homogeneous problem is

$$(18) \qquad \begin{cases} \operatorname{curl} \vec{\psi} = 0 \\ \vec{n} \times \vec{\psi} = 0 \quad \text{on } S_u \end{cases}$$

having as a general solution $\vec{\psi} = -\operatorname{grad} \chi$, with $\chi = 0$ on S_u. Condition (5) becomes

$$(19) \qquad 0 = \int_{\Omega} \vec{g} \cdot [-\operatorname{grad}] \chi d\Omega = \int_{\Omega} \chi \, [\operatorname{div}] \vec{g} \, d\Omega + \oint_{S} -\vec{g} \cdot \vec{n} \, \chi dS$$

hence being χ arbitrary inside Ω and on S_v it follows

$$(20) \qquad \operatorname{div} \vec{g} = 0 \quad \text{on } \Omega \quad \text{and} \quad \vec{n} \cdot \vec{g} = 0 \quad \text{on } S_v$$

These are the local compatibility conditions for $\vec{g}$, and therefore also the conservation laws for the physical quantity represented by g. As consequence of the fact that the problem (15 *a*) is unchanged by the substitution $\vec{\psi} \rightarrow \vec{\psi} - \operatorname{grad} \chi$ we see that the gauge invariance of problem (15 *a*) is tied with the conservation law of problem (14 *b*). Moreover the condition div curl $u \equiv 0$ is the Bianchi type identity for the equation (14.*b*)

4. ELASTIC FIELD. – For the small displacement theory the equations of equilibrium and those of compatibility are

$$(21\,a) \qquad [-\nabla_k] \, p^{hk} = f^h \qquad\qquad (21\,b) \qquad [\varepsilon^{hrl} \varepsilon^{ksm} \nabla_r \nabla_s] \, e_{em} = 0$$

being p^{hk} the stress tensor, e_{em} the strain tensor, f^h the volume force, ε^{hrl} the Ricci tensor and ∇_h the covariant derivative.

The general solutions of the corresponding homogeneous problems are respectively [4]

$$(22\,a)\qquad p^{hk} = \varepsilon^{hrl}\,\varepsilon^{ksm}\,\nabla_r\,\nabla_s\,\chi_{lm} \qquad\qquad (22\,b)\qquad e_{hk} = \left[\frac{1}{2} g_{hl}\,\nabla_k + \frac{1}{2}\,g_{kl}\nabla_h\right] u^l$$

being χ_{lm} a symmetric tensor, u_h the displacement vector. We note that the structure above indicated is here realized at least for the formal operators [5]. Are there local conservation laws for f^h? To this end we ask for the uniqueness solution of the adjoint homogeneous equation (22 *b*): $1/2\,(u_{h|k} + u_{k|h}) = 0$. No general solution of this equation exist but only the particular solution $\vec{u} = \vec{a} + \vec{b} \wedge r$ depending on two arbitrary vectors $\vec{a}$ and $\vec{b}$. Then we have no local conservation laws for f^h.

Are there identities for equation (21 *b*)? We ask for the uniqueness of the adjoint homogeneous equation $e^{hrl}\,\varepsilon^{ksm}\,\nabla_r\,\nabla_s\,\chi_{lm} = 0$. But this equation has exactly the same form as Eq. (21 *b*) and then its general solution [5] is: $\chi_{lm} = 1/2\,(\sigma_{l/m} + \sigma_{m/l})$ being σ_l an arbitrary vector. The formal adjoint of this formal operator is just $-\nabla_k$ then we have the identities

$$(23)\qquad -\nabla_k\,[\varepsilon^{hrl}\,\varepsilon^{hsm}\,\nabla_r\,\nabla_s]\,e_{lm} \equiv 0.$$

These are the well known Bianchi type identities for Saint Venant compatibility conditions. The transformation $\chi_{lm} \to \chi_{lm} + 1/2\,(\sigma_{l/m} + \sigma_{m/l})$ is a gauge transformation for the stress potential χ_{lm}.

5. ELECTROMAGNETIC FIELD. – The two sets of Maxwell equations for the electromagnetic field in matter are

$$(24\,a)\qquad [-\nabla_\alpha]\,f^{\alpha\beta} = \mathfrak{I}^\beta \qquad\qquad (24\,b)\qquad \left[\frac{1}{2}\,\varepsilon^{\alpha\beta\gamma\varrho}\,\nabla_\gamma\right] F_{\alpha\beta} = 0$$

being $f^{\alpha\beta}$ the electromagnetic tensor in matter, $F_{\alpha\beta}$ the electromagnetic tensor in vacuo and $\mathfrak{I}^\beta$ the current density.

The general solutions of the corresponding homogeneous problems are [6]

$$(25\,a)\qquad f^{\alpha\beta} = \left[\frac{1}{2}\,\varepsilon^{\alpha\beta\gamma\varrho}\,\nabla_\varrho\right]\psi_\gamma \qquad\qquad (25\,b)\qquad F_{\alpha\beta} = \left[\frac{1}{2}\,g_{\alpha\gamma}\,\nabla_\beta - \frac{1}{2}\,g_{\beta\gamma}\,\nabla_\alpha\right]\varphi^\gamma.$$

It is easy to see that the structure above indicated is here realized as far as formal operators [6] are concerned. We ask for local compatibility con-

(4) That the solution (22 *a*) is *general* was shown by a number of authors, see [8], showing the completeness of the stress potential χ_{lm} under specified derivability requirements. For the case that the body contains holes the solution (22 *a*) is no longer general and was extended by Gurtin [9].

(5) Solution (22 *b*) is general because Eq. (21 *b*) are necessary and *sufficient* conditions of existence for Eq. (22 *b*).

(6) These solutions are simply the extension to four dimensions of the corresponding general solution of the equation curl $\vec{v} = 0$: then the completeness can be shown using an analogous proof.

ditions for $\mathfrak{J}^{\beta}$: to this end we look for the uniqueness of the Eq. (25 *b*), that is we look for solutions of the homogeneous equation $1/2\,(\varphi_{\alpha/\beta} - \varphi_{\beta/\alpha}) = 0$. This equation has the general (6) solution $\varphi_\beta = -\nabla_\beta\,\chi$. Being $\mathfrak{R} = -\nabla_\beta$ it also will be $\tilde{\mathfrak{R}} = \nabla_\beta$ and the conservation law

$$\nabla_\beta\,\mathfrak{J}^{\beta} = 0 \tag{26}$$

and the Bianchi type identities $\nabla_\beta\,\nabla_\alpha f^{\alpha\beta} \equiv 0$ follow. We note that the non uniqueness of solution of (25 *b*) implies that the transformation $\varphi_\beta \rightarrow \varphi_\beta - \nabla_\beta\,\chi$ is a gauge transformation, a well known fact.

Are there identities for the second set (24 *b*)? We look for the uniqueness of the equation (25 *a*) and then for the general solution of equation $1/2\,[\varepsilon^{\alpha\beta\gamma\varrho}\nabla_\varrho]\psi_\gamma = 0$. The general (7) solution is $\psi_\gamma = -\nabla_\gamma\,\chi$ and then we have the identities

$$\nabla_\gamma\left[\frac{1}{2}\,\varepsilon^{\alpha\beta\gamma\varrho}\,\nabla_\gamma\right]F_{\alpha\beta} \equiv 0 \tag{27}$$

which are the Bianchi type identities for the second set of Maxwell equations.

6. CONCLUSION. – Known theorems of linear operators concerning existence and uniqueness of solutions combined with a general structure of linear field equations reveal in a new way the relation between gauge transformations and conservation laws for source terms and Bianchi type identities.

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(7) That solutions (25) are *general* can be seen using the three dimensional notation with curl, div, grad and using the results of vector field.