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Some Identities in Conformal Finsler Space

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Geometria differenziale. — *Some Identities in Conformal Finsler Space.* Nota di H. D. PANDE (*), presentata (**) dal Socio E. BOMPIANI.

SUNTO. — Identità soddisfatte dai tensori di curvatura nella geometria conforme degli spazi di Finsler.

1. INTRODUCTION.

In Finsler geometry there are mainly three kinds of covariant derivatives of a tensor, the connection coefficients are usually denoted by $\Gamma^{*i}_{jk}(x, \dot{x})$ and $G^i_{jk}(x, \dot{x})$, given by Cartan and Berwald respectively, which are symmetric in the indices j and k and are positively homogeneous of degree zero in their directional arguments. The three covariant derivatives are then the following:

Cartan's covariant derivative of, say, $T^i_j(x, \dot{x})$.

$$T^i_{j|k} = (\partial_k T^i_j) - (\dot{\partial}_m T^i_j)\, \Gamma^{*m}_{rk}\, \dot{x}^r + T^m_j\, \Gamma^{*i}_{mk} - T^i_m\, \Gamma^{*m}_{jk}, \tag{1.1}$$

$$T^i_j\big|_k = (\dot{\partial}_k T^i_j) + T^m_j A^i_{mk} - T^i_m A^m_{jk}, \tag{1.2}$$

where $A^i_{jh}(x, \dot{x}) = F(x, \dot{x})\, C^i_{jh}(x, \dot{x})$ which form a symmetric tensor [1]. Berwald's covariant derivative of, say, $T^i_j(x, \dot{x})$:

$$T^i_{j(k)} = (\partial_k T^i_j) - (\dot{\partial}_m T^i_j)\, G^m_k + T^m_j\, G^i_{mk} - T^i_m\, G^m_{jk}, \tag{1.3}$$

where

$$G^i_j(x, \dot{x}) \overset{\text{def}}{=} G^i_{mj}(x, \dot{x})\, \dot{x}^m = \Gamma^{*i}_{mj}(x, \dot{x})\, \dot{x}^m. \tag{1.4}$$

The covariant components of the unit vector along the direction of the element of support are given by

$$l_i(x, \dot{x}) \overset{\text{def}}{=} \dot{\partial}_i F(x, \dot{x}) \tag{1.5}$$

For covariant derivatives of the metric tensor $g_{ij}(x, \dot{x})$, we have $g_{ij|k} = 0 = g_{ij}\big|_k$ and $g_{ij(k)} = -2 A_{ijk|h}\, l^h$.

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(**) Nella seduta del 19 novembre 1968.

(1) $\partial_i = \partial/\partial x^i$ and $\dot{\partial}_i = \partial/\partial \dot{x}^i$.

(2) Numbers in brackets refer to the references at the end of the paper.

In this paper the notations used by Rund [1] have generally been adopted. With reference to the above covariant derivatives, we have the following commutation formulae [1, 2]:

(1.6) $$2\,X^i_{|[hk]} = K^i_{jhk}\,X^j - (\dot{\partial}_j X^i)\,K^j_{rhk}\,\dot{x}^r,$$

(1.7) $$2\,X^i|_{[hk]} = 2\,X^i|_{[h}\dot{\partial}_{k]}\,F + S^i_{jkh}\,X^j,$$

(1.8) $$X^i|_{h|k} - X^i_{|k}|_h = -\,P^i_{jkh}\,X^j + X^i|_j\,A^j_{hk|r}\,l^r + X^i_{|j}\,A^j_{hk}\,,$$

Where $X^i(x, \dot{x})$ is a vector field depending on position as well as on a directional argument, K^i_{jhk}, S^i_{jhk} and P^i_{jhk} are the curvature tensors given by

(1.9) $$K^i_{jhk}(x, \dot{x}) = (\partial_k \Gamma^{*i}_{jh} - \dot{\partial}_m \Gamma^{*i}_{jh}\,G^m_k) - (\partial_h \Gamma^{*i}_{jk} - \dot{\partial}_m \Gamma^{*i}_{jk}\,G^m_h) + \Gamma^{*i}_{mk}\,\Gamma^{*m}_{jh} - \Gamma^{*i}_{mh}\,\Gamma^{*m}_{jk},$$

(1.10) $$S^i_{jhk}(x, \dot{x}) = A^i_{mh}\,A^m_{jk} - A^i_{mk}\,A^m_{jh},$$

and

(1.11) $$P^i_{jhk}(x, \dot{x}) = F\dot{\partial}_k \Gamma^{*i}_{jh} + A^i_{jm}\,A^m_{hk|r}\,l^r - A^i_{jk|h},$$

respectively.

2. CONFORMAL CORRESPONDENCE.

Let us consider the conformal transformation

(2.1) $$\bar{g}_{ij} = e^{2\sigma}\,g_{ij},$$

where $\sigma = \sigma(x)$. In consequence of (2.1), we have the following entities of the conformal Finsler space [3, 4]

(2.2) $$\bar{F}(x, \dot{x}) = e^{\sigma}\,F(x, \dot{x}),$$

(2.3) $$\bar{G}^i(x, \dot{x}) = G^i(x, \dot{x}) - \sigma_m\,B^{im}(x, \dot{x}),$$

(2.4) $$\bar{\dot{x}}^i = \dot{x}^i,$$

(2.5) $$\bar{l}^i = \bar{e}^{\sigma}\,l^i$$

(2.6) $$\bar{\Gamma}^{*i}_{jk}(x, \dot{x}) = \Gamma^{*i}_{jk}(x, \dot{x}) + U^i_{jk}(x, \dot{x}),$$

where $\sigma_m = \partial_m \sigma$, $B^{im}(x, \dot{x}) \overset{\text{def}}{=\!=} \frac{1}{2} F^2 g^{im} - \dot{x}^i \dot{x}^m$,

$$U^i_{jk}(x, \dot{x}) \overset{\text{def}}{=\!=} 2\,\sigma_{(j}\delta^i_{k)} - \sigma_m\,\{g^{im}\,g_{jk} - 2\,C^i_{r\,(j}\dot{\partial}_{k)}\,B^{rm} + g^{ir}\,C_{jks}\,\dot{\partial}_r\,B^{sm}\}.$$

3. Identities satisfied by curvature tensors in conformal Finsler space.

The curvature tensors $P^i_{jkh}(x, \dot{x})$ and $S^i_{jkh}(x, \dot{x})$ transform under the conformal change (2.1) as follows [3, 4]:

$$
\begin{aligned}
(3.1)\qquad \bar{P}^i_{jkh}(x, \dot{x}) &= e^\sigma [P^i_{jkh} + \sigma_j A^i_{kh} - 3\sigma_m g^{im} A_{jkh} - \sigma_r l^r S^i_{jkh} + \\
&+ \sigma_n \{(\dot{\partial}_r A^i_{kh})(\dot{\partial}_j B^{rn}) - g^{im}(\dot{\partial}_r A_{jkh})(\dot{\partial}_m B^{rn})\} - \\
&- F g^{sn} \sigma_n \{A^i_{mk}(\dot{\partial}_s A^m_{jh}) - A^m_{jk}(\dot{\partial}_s A^i_{mh})\} + A^r_{kh} U^i_{rj} - 2 A^i_{r(h} U^r_{k)j} + A_{rkh} g^{im} U^r_{jm} + \\
&+ 2 g^{im} A_{jr(h} U^r_{k)m} - l^r \{2 A^i_{m(k} A^s_{h)j} U^m_{sr} + U^s_{hr} S^i_{jsk} - A^i_{mk} A^m_{sh} U^s_{jr} - A^m_{jk} A^s_{mh} U^i_{sr}\}]
\end{aligned}
$$

and

$$(3.2)\qquad \bar{S}^i_{rjk}(x, \dot{x}) = e^{2\sigma} S^i_{rjk}.$$

We have the following theorems:

THEOREM 3.1. *The curvature tensor* $\bar{P}^i_{jkh}(x, \dot{x})$ *satisfies the following identities* (3):

$$(3.3)\qquad \bar{P}^i_{[jkh]} = e^\sigma P^i_{[jkh]},$$

and

$$(3.4)\qquad \bar{P}_{[j\,\boxed{m}\,kh]} = e^{3\sigma} P_{[j\,\boxed{m}\,kh]},$$

where

$$(3.5)\qquad \bar{P}_{jmkh}(x, \dot{x}) = \bar{g}_{im} \bar{P}^i_{jkh}.$$

Proof. Using the symmetric and homogeneity properties of functions $A^i_{jk}(x, \dot{x})$, $U^i_{jk}(x, \dot{x})$ and $B^{ij}(x, \dot{x})$, we obtain

$$(3.6)\qquad S^i_{[jkh]} = 0,$$

$$(3.7)\qquad A^i_{m[k} A^m_{h\,\boxed{s}} U^s_{j]r} = 0,$$

and

$$(3.8)\qquad U^s_{r[h} S^i_{j\,\boxed{s}\,k]} = 0,$$

Taking the skew-symmetric part with respect to indices j, k and h in (3.1) and using the above relations, we get the result (3.3).

(3) The round brackets denote the symmetric part, i. e.

$$2\, T_{(i\,\boxed{j}\,k)} = T_{ijk} + T_{kji},$$

the square brackets denote the skew – symmetric part, i. e.

$$2\, T_{[i\,\boxed{j}\,k]} = T_{ijk} - T_{kji},$$

and the indices enclosed in a rectangle are excluded from a symmetric and skew – symmetric parts.

Again, we obtain from (3.5)

$$\overline{\mathrm{P}}_{[j\,|\overline{m}|\,kh]} = \bar{g}_{im}\, \overline{\mathrm{P}}^{i}_{[jkh]}. \tag{3.9}$$

With the help of equations (2.1), (3.3) and (3.9), we get (3.4).

THEOREM 3.2. *We have*

$$\overline{\mathrm{P}}^{i}_{j[kh]} = e^{\sigma}\, [\mathrm{P}^{i}_{j[kh]} + l^{r} \{ 2\, \sigma_r\, \mathrm{S}^{i}_{jhk} + (\dot{\partial}_m\, \mathrm{S}^{i}_{jhk})\, (\dot{\partial}_r\, \mathrm{B}^{mn})\, \sigma_n + \mathrm{S}^{m}_{jhk}\, \mathrm{U}^{i}_{mr} - \\ - \mathrm{S}^{i}_{mhk}\, \mathrm{U}^{m}_{jr} - \mathrm{S}^{i}_{jmk}\, \mathrm{U}^{m}_{hr} - \mathrm{S}^{i}_{jhm}\, \mathrm{U}^{m}_{kr} \}]. \tag{3.10}$$

Proof. The curvature tensor $\mathrm{P}^{i}_{jkh}(x\,,\dot{x})$ satisfies the following identity:

$$2\, \mathrm{P}^{i}_{j[kh]} = \mathrm{S}^{i}_{jhk\,|\,r}\, l^{r}. \tag{3.11}$$

Under the conformal change (2.1) the above identity transforms to

$$2\, \overline{\mathrm{P}}^{i}_{j[kh]} = \overline{\mathrm{S}}^{i}_{jhk\,\overline{|}\,r}\, \bar{l}^{r}, \tag{3.12}$$

where the symbol $\overline{|}$ followed by an index denotes the Cartan's first covariant derivative for the connection coefficients $\overline{\Gamma}^{*i}_{jk}(x\,,\dot{x})$, of the conformal Finsler space [5], i.e. we have

$$\overline{\mathrm{S}}^{i}_{jhk\overline{|}r} = \partial_r \overline{\mathrm{S}}^{i}_{jhk} - (\dot{\partial}_m\, \overline{\mathrm{S}}^{i}_{jhk})\, \overline{\mathrm{G}}^{m}_{r} + \overline{\mathrm{S}}^{m}_{jhk}\, \overline{\Gamma}^{*i}_{mr} - \overline{\mathrm{S}}^{i}_{mhk}\, \overline{\Gamma}^{*m}_{jr} - \overline{\mathrm{S}}^{i}_{jmk}\, \overline{\Gamma}^{*m}_{hr} - \overline{\mathrm{S}}^{i}_{jhm}\, \overline{\Gamma}^{*m}_{kr}, \tag{3.13}$$

Using equations (2.3), (2.5), (2.6), (3.2) and (3.13) in (3.12), we get the identity (3.10),

THEOREM 3.3. *The identity satisfied by the curvature tensor* $\overline{\mathrm{P}}_{ijkh}(x\,,\dot{x})$ *reads as follows*:

$$\{ \overline{\mathrm{P}}_{ijkh} - \overline{\mathrm{P}}_{khij} \}\, \bar{\dot{x}}^{i} = e^{3\sigma}\, [\{ \mathrm{P}_{ijkh} - \mathrm{P}_{khij} \}\, \dot{x}^{i} + \{ 3\sigma_i\, \mathrm{A}_{hjk} + \\ + (\dot{\partial}_r\, \mathrm{A}_{hjk})\, (\dot{\partial}_i\, \mathrm{B}^{rm})\, \sigma_m - \mathrm{A}_{mjk}\, \mathrm{U}^{m}_{hi} - \mathrm{A}_{hmk}\, \mathrm{U}^{m}_{ji} - \mathrm{A}_{hjm}\, \mathrm{U}^{m}_{ki} \}\, \dot{x}^{i}]. \tag{3.14}$$

Proof. In view of the identity $(\mathrm{P}_{ijkh} - \mathrm{P}_{khij})\, \dot{x}^{i} = \mathrm{A}_{hjk\,|\,i}\, \dot{x}^{i}$ [4], the curvature tensor $\overline{\mathrm{P}}_{ijkh}(x\,,\dot{x})$ satisfies

$$(\overline{\mathrm{P}}_{ijkh} - \overline{\mathrm{P}}_{khij})\, \bar{\dot{x}}^{i} = \bar{\mathrm{A}}_{hjk\overline{|}i}\, \bar{\dot{x}}^{i}, \tag{3.15}$$

With the help of equations (2.3), (2.4), (2.6) and the relation $\bar{\mathrm{A}}_{ijk}(x\,,\dot{x}) = e^{3\sigma}\, \mathrm{A}_{ijk}$, in (3.15), we obtain (3.14).

THEOREM 3.4. *The curvature tensor* $\overline{\mathrm{S}}^{i}_{jhk}(x\,,\dot{x})$ *satisfies the following Bianchi identity*:

$$\overline{\mathrm{S}}^{i}_{j[hk\,\overline{|}\,m]}\, \dot{x}^{k} = e^{3\sigma}\, \mathrm{S}^{i}_{j[hk\,|\,m]}\, \dot{x}^{k}, \tag{3.16}$$

where the symbol $\overline{|}$ followed by an index denotes the Cartan's second type of covariant derivative in the conformal Finsler space [6].

Proof. We have the identity $\mathrm{S}^{i}_{j[hk\,|\,m]}\, \dot{x}^{k} = \mathrm{F}\, \mathrm{S}^{i}_{jmh}$ [4]. Using equations (2.2) and (3.2) we can easily transform this identity to the form given by equation (3.16) under the conformal change (2.1).

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