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Daniell's Method in the Theory of the Aumann-Hukuhara Integral of Set-Valued Functions

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Teoria dell'integrale. — *Daniell's Method in the Theory of the Aumann–Hukuhara Integral of Set–Valued Functions.* Nota di FRANCESCO S. DE BLASI e ANDRZEJ LASOTA, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Viene fatto un confronto tra due definizioni di integrale per funzioni multivoche proposte da Aumann e da Hukuhara.

The theory of integrals of set-valued functions goes back to J. Aumann [1]. It has been studied in connection with differential equations, optimal control theory [6], [8] and classical problems of functional analysis such as the theory of extremal points [2] and Liapunoff's theorem [2], [5], [6]. The reader can find further references, new results and interesting open questions in [3].

A new easy approach to the definition of the integral of set-valued functions, based on the ideas of Riemann and Daniell, was recently given by M. Hukuhara [4]. The Hukuhara integral is defined on the space X_B of all integrable (see definition below) set-valued functions $t \to F(t)$ with convex compact values $F(t)$. Our purpose is to prove that the restriction of Aumann's integral to the space X_B coincides with the integral of Hukuhara.

The space X_B with suitable defined distance is a complete metric space. Using the Hukuhara technique in the first part of the present note we shall show that his integral may be immediately defined as the continuous extension of an uniformly continuous function given on a dense subset $X_S \subset X_B$. This short construction also shows that the space X_B with Hukuhara's integral is a natural generalisation of the space L^1 of all single-valued Lebesgue integrable functions. The proof that the theory of Aumann's integral in the space X_B is equivalent to that of Hukuhara is given in the part 2.

1. Denote by (H, r) the metric space of all nonempty convex compact subsets of R^k where the metric function r is given by the Hausdorff distance. Let D be a measurable subset of R^l such that the Lebesgue measure $0 < \mu(D) < \infty$. A function $F: D \to H$ is said to be measurable if for each $C \in H$ the set $\{t : F(t) \cap C \neq \varnothing\}$ is Lebesgue measurable [7]. A measurable function $F: D \to H$ is called integrable if the single-valued function $|F(t)| = r(F(t), 0)$ is Lebesgue integrable. The set of all integrable functions $F: D \to H$ will be denoted by X_B. For $F, G \in X_B$ we put

$$\mathrm{Dist}(F, G) = \int_D r(F(t), G(t))\, dt .$$

(*) Nella seduta del 19 novembre 1968.

It is easy to see that the space (X_B, Dist) is a metric space if we as usual identify functions different only on a set of measure zero.

THEOREM 1. *The space* (X_B, Dist) *is complete.*

The proof is quite analogous to that for single-valued functions. In fact let us suppose that $\text{Dist}(F_n, F_m) \to 0$. By the standard argument we can choose a subsequence F_{i_n} with the following property: for each $\varepsilon > 0$ there is a subset $D_\varepsilon \subset D$ such that $\mu(D \setminus D_\varepsilon) < \varepsilon$ and $r(F_{i_n}(t), F_{i_m}(t)) \to 0$ uniformly on D_ε. Since the space (H, r) is complete there exists a set $F(t) \in H$ defined for almost all $t \in D$ such that $r(F_{i_n}(t), F(t)) \to 0$ a.e. on D. The function $F : D \to H$ as a limit of measurable functions is measurable too. Fix $\varepsilon > 0$, $\delta > 0$. We have

$$\int_{D_\varepsilon} r(F_n(t), F_m(t))\, dt \leq \delta$$

for sufficiently large n $(n \geq n_\delta)$. So we can put $m = i_s$ and pass to the limit with $s \to \infty$ and then with $\varepsilon \to 0$. We obtain

$$\int_{D_\varepsilon} r(F_n(t), F(t))\, dt \leq \delta$$

for $n \geq n_\delta$. From this it follows simultaneously that F is integrable and $\text{Dist}(F_n, F) \to 0$.

Let $X_S \subset X_B$ be the subspace of all step functions, i.e., functions given by the formulae

$$(1) \qquad F(t) = \sum_{i=1}^{n} \chi_{D_i}(t)\, C_i \quad , \quad \bigcup_{i=1}^{n} D_i = D \quad , \quad D_i \cap D_j = \varnothing \qquad (i \neq j)$$

where χ_{D_i} stands for the characteristic function of a measurable set D_i and $C_i \in H$. The Hukuhara integral of the function (1) is given by

$$(2) \qquad I(F) = \sum_{i=1}^{n} \mu(D_i)\, C_i \,.$$

Let us observe that

$$(3) \qquad r(I(F), I(G)) \leq \text{Dist}(F, G)$$

for $F, G \in X_S$. So $I : X_S \to H$ is an uniformly continuous function and can be uniquely extended to the whole X_B, since X_S is dense in X_B. To prove that let us remember the following (see [4] p. 219).

LEMMA 1. (Hukuhara). *For each* $\varepsilon > 0$ *and* $F \in X_B$ *such that* $|F(t)| \leq c$ $(c = \text{const})$ *there exists a function* $F' \in X_S$ *such that* $\text{Dist}(F', F) < \varepsilon$ *and* $|F'(t)| \leq |F(t)| + 1$.

THEOREM 2. *The set* X_S *is dense in* (X_B, Dist). *Moreover, for each* $F \in X_B$ *there exists a sequence* $\{F_n\}$ *in* X_S *such that* $\text{Dist}(F_n, F) \to 0$ *and* $|F_n(t)| \leq |F(t)| + 1$.

Setting $F'_n(t) = F(t) \cap \{x \in R^k : |x| \leq n\}$ we have

$$\mathrm{Dist}(F'_n, F) \leq \int_{D_n} r(F'_n(t), F(t))\, dt \leq 2 \int_{D_n} |F(t)|\, dt = \varepsilon_n$$

where $D_n = \{t : |F(t)| \geq n\}$. Since the function $|F|$ is integrable, $\varepsilon_n \to 0$. Evidently $|F'_n(t)| \leq |F(t)|$ and to end the proof it is sufficient to choose by lemma 1 for every F'_n a function F_n such that $\mathrm{Dist}(F_n, F'_n) \leq 1/n$.

From theorems 1, 2 and inequality (3) it follows immediately the following extension theorem.

THEOREM 3. *There exists exactly one continuous function* $I : X_B \to H$ *which is equal to* (2) *on* X_S. *The function* I *satisfies on* X_B *inequality* (3).

In fact the extension of a uniformly continuous function from a dense set exists and is unique and the extension of a Lipschitz function is a Lipschitz function with the same constant (in this case equal 1).

2. Denote by A the family of all nonempty subsets of R^k. We write $\lim A_n = A$ if $\limsup A_n = \liminf A_n = A$. By definition $x \in \liminf A_n$ if and only if every neighborhood of x intersects all the A_n with sufficiently high n and $x \in \limsup A_n$ if and only if every neighborhood intersects infinitely many A_n. It is easy to see that for $A_n, A \in H$ $(\sup |A_n| < \infty)$

(5) $\lim r(A_n, A) = 0$ is equivalent to $\lim A_n = A$.

Let X be the space of all functions $F : D \to A$. For $F \in X$ the Aumann integral is given by

$$J(F) = \left\{ \int_D f(t)\, dt : f \text{ is integrable }, f(t) \in F(t) \text{ a.e.} \right\}.$$

Denote the restriction of J to X_B by J_B. We shall use the following properties of J_B:

(*i*) If $F_C(t) \equiv C$ $(C \in H)$ then $J_B(F_C) = \mu(D)\, C$.

(*ii*) If $D_1, \cdots, D_n$ are disjoint measurable subsets of D and $D = \bigcup_{i=1}^{n} D_i$ then

$$J_B(F) = \sum_{i=1}^{n} J_B(\chi_{D_i} F)$$

for every $F \in X_B$.

(*iii*) If $\lim F_n(t) = F(t)$ a.e. $(F_n, F \in X_B)$ and if there exists a single-valued integrable function m such that $|F_n(t)| \leq m(t)$ $(t \in D)$, then $\lim J_B(F_n) = J_B(F)$.

The proof of (*iii*) is given by Aumann (see [1] th. 5). Property (*ii*) follows immediately from the definition of J_B. As for (*i*) it is easy also to see that $\mu(D)\, C \subset J_B(F_C)$. To prove that $J_B(F_C) \subset \mu(D)\, C$ it is sufficient to apply the integral form of the mean value theorem

$$\int_D f(t)\, dt \subset \mu(D)\, \overline{co}\, \{f(t) : t \in D\}.$$

From (i) and (ii) it follows that for step function (1) the integral J_B is given by

$$J_B(F) = \sum_{i=1}^{n} \mu(D_i)\, C_i \,. \tag{6}$$

So both the integrals I and J_B are equal on X_S, i.e.

$$J_B(F) = I(F) \qquad F \in X_S \tag{7}$$

and consequently

$$r(J_B(F), J_B(G)) \leq \text{Dist}(F, G) \qquad F, G \in X_S .$$

Suppose now that F, G are arbitrary functions in X_B. By theorem 2 we can choose two sequences $\{F_n\}, \{G_n\}$ such that

$$\text{Dist}(F_n, F) \to 0 \quad , \quad \text{Dist}(G_n, G) \to 0 \tag{8}$$

and

$$|F_n(t)| \leq |F(t)| + 1 \,, |G_n(t)| = |G(t)| + 1 \tag{9}$$

Using Egoroff theorem and passing to subsequences we can suppose that

$$r(F_n(t), F(t)) \to 0 \quad , \quad r(G_n(t), G(t)) \to 0 \quad \text{a.e.} \tag{10}$$

We have

$$r(J_B(F_n), J_B(G_n)) \leq \text{Dist}(F_n, G_n) \tag{11}$$

From (8) it follows that the right-hand side of inequality (11) converges to Dist (F, G). Inqualities (9) imply

$$|J_B(F_n)| \leq \int_D |F(t)|\, dt + \mu(D) \quad , \quad |J_B(G_n)| \leq \int_D |G(t)|\, dt + \mu(D). \tag{12}$$

By using (10), (9), (iii), (12) and (5) we see that the left-hand side of (11) converges to $r(J_B(F), J_B(G))$. So we obtain

$$r(J_B(F), J_B(G)) \leq \text{Dist}(F, G) \qquad F, G \in X_B .$$

From this it follows that the Aumann integral J_B is a continuous function on X_B and consequently by (7), continuity of I and theorem 2 we have

THEOREM 4. *On the space* X_B *the Aumann integral is equal to that of Hukuhara, i.e.*

$$J_B(F) = I(F) \qquad F \in X_B .$$

ADDED IN PROOF. While this paper was in print we became acquainted, through the courtesy of Dr. G. S. Goodman, of a recent paper of G. Debreu « Integration of correspondences, Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, vol. II, part 1, 351–72, Univ. of Cal. Press, Berkeley & Los Angeles, 1967» where among many other interesting results the equivalence of Aumann and Hukuhara integrals is established by using an argument which is different from ours (and less direct).

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