Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

A. Charnes, William W. Cooper, Michael J. L. Kirby

Optimal Decision Rules in Conditional Probabilistic Programming

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **45** (1968), n.5, p. 231–235.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1968_8_45_5_231_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1968.

Ricerca operativa. — *Optimal Decision Rules in Conditional Probabilistic Programming* [1]. Nota [*] di A. CHARNES [2], WILLIAM W. COOPER [3], e MICHAEL J.L. KIRBY [4], presentata dal Socio B. SEGRE.

RIASSUNTO. — Nella presente Nota si sviluppa una teoria unificata della programmazione probabilistica. Questa, nell'adoperare dei vincoli probabilistici condizionali, porge una caratterizzazione delle classi ottimali di regole stocastiche di decisione. In particolare, vien stabilita l'ottimalità delle regole lineari discretizzate di decisione per la minimizzazione del valore assunto da una funzione concava di variabili di decisioni stocastiche.

INTRODUCTION.

In [1, 2, 3, 4, 5] it has been shown how the study of major forms of probabilistic programming can be carried out from the point of view of chance constrained programming. Past results on the optimality of stochastic decision rules, in computation of specific examples, etc., have depended on sophisticated constructs, elaborate analyses, ingenious devices and in severe restrictions not only on admissible classes of stochastic decision rules but also on the functionals to be optimized. [See references 1–15]. In the following we focus on problems with the seemingly more complicated conditional chance constraints and develop an analytical tool which unifies in form the informational bases of different varieties of probabilistic programming and, at the same time, provides a direct characterization of optimal classes of stochastic decision rules. In particular, the optimality of piecewise linear decision rules is established for the general objective, to minimize the expected value of a concave function of the stochastic decision variables. Characterizations are also developed for the optimal stochastic decision rules for the problem of minimizing the expected value of a general convex (differentiable) function of the stochastic variables. In addition, necessary conditions are developed for other general classes of objective functions. Corollaries are general results for problem with (possibly) non-linear functionals and linear programming under uncertainty constraints. In fact, the results are extended to heretofore never considered mixed systems of conditional chance-constraints and linear programming under uncertainty constraints.

(*) Pervenuta all'Accademia il 30 luglio 1968.
(1) To Dr. Gunter Karl von Noorden whose thought and hand restored the sight of A. Charnes.
(2) Northwestern University and University of Texas.
(3) Carnegie–Mellon University.
(4) Dalhousie University.

ANALYTICAL RESULTS.

The general n–stage chance-constrained problem with linear conditional chance-constraints and constant structural matrix may be represented as

$$\text{(1)} \qquad \text{Min EH}\,(\cdots, x_j\,(b_1,\cdots, b_{j-1}),\cdots)$$

$$\text{s.t.}\quad \bar{\text{P}}\left(\sum_{j=1}^{i} \text{A}_{ij}\,x_j\,(b_1,\cdots, b_{j-1}) \leq b_i\right) \geq \bar{\alpha}_i \qquad i = 1,\cdots, m$$

$$x_j\,(b_1,\cdots, b_{j-1}) \geq 0\,, \qquad j = 1,\cdots, n$$

where $b_i^{\text{T}} \equiv (b_{i1},\cdots, b_{im})$, $\bar{\text{P}}$ denotes probability conditional on $b_1,\cdots, b_{i-1}$, $\bar{\alpha}_i^{\text{T}} \equiv (\bar{\alpha}_{i1},\cdots, \bar{\alpha}_{im_i})$ is a vector of functions of $b_1,\cdots, b_{i-1}$ with all $\bar{\alpha}_{ij}$ in $[0, 1]$, the A_{ij} are constant matrices and E is the expectation operator.

As in [4], these chance-constraints may be inverted to give the equivalent linear inequality constraints:

$$\text{(2)} \qquad \sum_{j=1}^{i} \text{A}_{ij}\,x_j\,(b_1,\cdots, b_{j-1}) \leq \bar{\text{F}}_i^{-1}(\bar{\alpha}_i) \qquad i = 1,\cdots, m$$

$$x_j\,(b_1,\cdots, b_{j-1}) \geq 0 \qquad j = 1,\cdots, n$$

where $\bar{\text{F}}_i^{-1}(\bar{\alpha}_i)$ is the vector of marginal fractiles of b_i conditional on $b_1,\cdots, b_{i-1}$. Further, in equality form (2) becomes:

$$\text{(3)} \qquad \sum_{j=1}^{i} \text{A}_{ij}\,x_j\,(b_1,\cdots, b_{j-1}) + s_i\,(b_1,\cdots, b_{i-1}) = \bar{\text{F}}_i^{-1}(\bar{\alpha}_i) \qquad i = 1,\cdots, m$$

$$x_j\,,\, s_i \geq 0 \qquad i = 1,\cdots, m\,;\; j = 1,\cdots, n.$$

The following system renders in a single form both the previous systems and new mixed systems of conditional chance-constraints and linear programming under uncertainty constraints:

$$\text{(4)} \qquad \min \text{EH}\,(\lambda^1,\cdots, \lambda^n)$$

$$\text{s.t.}\quad \sum_{j=1}^{i} \text{P}_{ij}\,\lambda^j\,(b_1,\cdots, b_{j-1}) = g_i\,(b_1,\cdots, b_{i-1}) \qquad i = 1,\cdots, m$$

$$\lambda^j\,(b_1,\cdots, b_{j-1}) \geq 0 \qquad j = 1,\cdots, n.$$

We assume henceforth that this system is consistent for every $(b_1,\cdots, b_n)$ and that the columns of the P_{ii} have the Opposite Sign Property. [See 13 and 16]. In analyzing the system (4) we assume the information pattern to be the following: The λ^i vector is determined after λ^j, g_j, $j = 1,\cdots, i-1$ and $g_i\,(b_1,\cdots, b_{i-1})$ are known but before all other λ_j, g_j and $b_i,\cdots, b_m$ are known. We call vectors λ^i which are determined recursively in this way "informationally feasible" vectors.

LEMMA 1: Let $L^{i\#}$ denote the vector with matrix components each of whose entries is a left inverse of one of the N_i submatrices of P_{ii} which consist of linearly independent columns.

Let $\mu^i \equiv \mu^i(b_1, \cdots, b_{i-1}) \equiv (\mu_1^i(b_1, \cdots, b_{i-1}), \cdots, \mu_{N_i}^i(b_1, \cdots, b_{i-1})) \geq 0$

be a conformable vector of scalars with $\mu^{iT} e = 1$, where e is a vector with all entries unity.

Let $K^i \equiv \mu^{iT} L^{i\#}$.

Then the general informationally feasible decision rule for (4) is defined recursively by:

$$\lambda^i = K^i \left(g_i - \sum_{j=1}^{i-1} P_{ij} \lambda^j \right) \quad \text{where} \quad \lambda^i \geq 0, \qquad i = 1, \cdots, n. \tag{5}$$

LEMMA 2: The set of informationally feasible decision rules for (4) is convex and spanned by the finite number of extreme points which may be represented recursively by

$$\hat{\lambda}^i = E^i \left(g_i - \sum_{j=1}^{i-1} P_{ij} \hat{\lambda}^j \right), \quad \text{where} \quad \hat{\lambda}^i \geq 0, \qquad i = 1, \cdots, n \tag{6}$$

and E^i denotes a component of $L^{i\#}$.

Thus each extreme point decision rule is defined by a sequence of left-inverses for which the g_i satisfy a system of linear inequalities defined by these left-inverses.

LEMMA 3: There exists a finite disjoint decomposition of the domain of the $b_1, \cdots, b_n$ on each set of which only a finite number of extreme point decision rules hold.

THEOREM 1: *If* $H(\lambda^1, \cdots, \lambda^n)$ *is concave, then there exists a vector of optimal decision rules for* (4) *which is piece-wise linear in the* g_i. *The pieces correspond to extreme point rules, thus the optimal rules involve shifting between a finite number of linear decision rules.*

COROLLARY 1: Piece-wise linear decision rules are optimal for problems with (possibly mixed) systems of conditional chance constraints and linear programming under uncertainty constraints.

THEOREM 2: *If* $H(\lambda^1, \cdots, \lambda^n)$ *is convex and differentiable, and the convex set of decision rules has an interior point, then a vector of decision rules* $(\lambda^1, \cdots, \lambda^n)$ *is optimal if and only if*

$$\sum_{i=l}^{n} \frac{\partial H}{\partial \lambda_\alpha^i} \frac{\partial \lambda_\alpha^i}{\partial \mu_\beta^l} = \sum_{i=l}^{n} \nu_{i\alpha} \frac{\partial \lambda_\alpha^i}{\partial \mu_\beta^l} \quad \textit{for some} \quad \nu_{i\alpha} \geq 0 \tag{7}$$

and $\nu_{i\alpha} \lambda_\alpha^i = 0$.

THEOREM 3: *If* H *is differentiable and the convex set of decision rules has an interior point, then* (7) *is necessary for optimality.*

CONCLUSIONS.

The above lemmas and theorems provide a new route for the study of probabilistic programming, including chance-constrained programming, linear programming under uncertainty, etc. Extensions of the above results and specializations permitting sharper conclusions will be developed in forthcoming publications. In particular, further developments characterizing the constraint set and the optimal decision rules from both geometric and algebraic viewpoints will be given. Further results involving other constraint sets and information patterns will be presented. In the meantime the above development provides heretofore missing elements for unifying various approaches to probabilistic programming.

REFERENCES.

[1] A. CHARNES, W. W. COOPER and G. L. THOMPSON, *Constrained generalized medians and linear programming under uncertainty*. O.N.R. Research Memorandum No. 41, Evanston, Ill., Northwestern University, The Technological Institute, July 1961.

[2] A. CHARNES and W. W. COOPER, *Critical Path Analyses via Chance Constrained and Stochastic Programming*, « Operations Research », 12, No. 3, May–June 1964, pp. 460–470.

[3] A. CHARNES, M. J. L. KIRBY and W. M. RAIKE, *Solution Theorems in Probabilistic Programming*, « Journal of Mathematical Analyses and Applications », Vol. 20, No. 3, Dec. 1967, pp. 565–582.

[4] A. CHARNES and M. J. L. KIRBY, *Optimal decision rules for the E–model of chance–constrained programming*, « Cahiers Centre Etudes Recherche Operationnelle », *8*, 5–44 (1966).

[5] A. CHARNES and M. J. L. KIRBY, *Optimal decision rules for the triangular chance–constrained programming problem*. Systems Research Memorandum No. 115, Evaston, Ill., Northwestern University, 1965, « Management Sci. » (forthcoming).

[6] R. WETS, *Programming under uncertainty: The complete problem*, « Z. Wharscheinlichkeitstheorie Verw. Gebiete », *4*, 316–339 (1966).

[7] R. WETS, *Programming under uncertainty: The equivalent convex problem*, « J. SIAM Appl. Math. », *14*, 89–105 (1966).

[8] D. W. WALKUP and R. WETS, *Stochastic programs with recourse*. Seattle Wash., Boeing Scientific Research Laboratories, Document D1–82–0551, July, 1966.

[9] G. B. DANTZIG, *Linear programming under uncertainty*, « Management Sci. », *1*, 197–206 (1955).

[10] A. CHARNES, W. W. COOPER and G. H. SYMONDS, *Cost Horizons and Certainty Equivalents: An Approach to Stochastic Programming of Heating Oil*, « Management Science », *4*, No. 3, April 1958, pp. 235–263.

[11] A. CHARNES, W. W. COOPER and G. L. THOMPSON, *Constrained generalized medians and hypermedians as deterministic equivalents for two–stage linear programs under uncertainty*, « Management Sci. », *12*, 83–112 (1965).

[12] M. J. L. KIRBY, *The Current State of Chance–Constrained Programming*, to appear in the *Proceedings of the 1967 International Symposium on Mathematical Programming*.

[13] A. CHARNES and W. W. COOPER, *Management Models and Industrial Applications of Linear Programming*, 2 vols. Wiley, New York 1961.

[14] A. CHARNES and M. J. L. KIRBY, *Some special P–Models in chance–constrained programming*, Systems Research Memorandum No. 135, Evanston, Ill., Northwestern University, August, 1965, « Management Sci. », Vol. 14, No. 3, Nov. 1967, pp. 183–95.

[15] A. C. WILLIAMS, *On stochastic linear programming*, « J. SIAM Appl. Math. », *13*, 927–940 (1965).

[16] A. CHARNES, K. O. KORTANEK and W. M. RAIKE, *Extreme-point solutions in mathematical programming: An opposite-sign algorithm*, « Systems Research Memorandum », No. 129, June 1965, Northwestern University.

[17] A. CHARNES and W. W. COOPER, *Deterministic equivalents for optimizing and satisfying under chance constraints*, « Operations Res. », *11*, 18–39 (1963).

[18] R. M. VAN SLYKE and R. WETS, *Programming under uncertainty and stochastic optimal control*, Seattle, Wash., « Boeing Scientific Research Laboratories », Document D1-82-0429, April 1967.

[19] A. CHARNES and W. W. COOPER, *Chance Constrained Programming*, « Management Science », 6, No. 1, October 1959, pp. 73–80.