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A Generalized Picone Identity

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Matematica. — *A Generalized Picone Identity.* Nota di KURT KREITH, presentata (*) dal Socio M. PICONE.

RIASSUNTO. — Questa Nota presenta concise dimostrazioni di alcuni teoremi di confronto relativi ad equazioni ellittiche autoaggiunte e una dimostrazione elementare di un teorema variazionale concernente il primo autovalore di un problema al contorno ellittico. Le dimostrazioni del teorema di confronto basate su un'identità integrale (teorema 1°) generalizzano la ben nota identità di Picone per le equazioni differenziali ordinarie. L'applicazione di questa identità rende possibile una semplificazione sostanziale delle dimostrazioni di teoremi di confronto per equazioni ellittiche che furono presentate nei *Proceedings of the American Mathematical Society* da P. Hartman e A. Winter (1955), dall'autore (1963) e da C. Swanson e C. Clark (1965).

L'autore ha recentemente appreso che questa identità integrale (teorema 1°) e le sue principali applicazioni (teorema 2°) furono presentate a questa Accademia dal Professor M. Picone nel 1911. L'opera del Professor Picone in tale argomento è stata disgraziatamente trascurata dall'autore e da altri contemporanei scrittori in questo campo, la cui opera sarebbe stata molto avvantaggiata dalla conoscenza di quelle antiche interessantissime note lincee.

Picone's identity deals with functions $u(x)$ and $v(x)$ which are, respectively, solutions of Sturm-Liouville equations

$$(au')' - cu = 0$$

$$(\alpha v')' - \gamma v = 0$$

on an interval $[x_1, x_2]$. If $v(x) \neq 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{u}{v}(au'v - \alpha uv')\right] = (c - \gamma)u^2 + (a - \alpha)u'^2 + \alpha\left(u' - u\frac{v'}{v}\right)^2 \tag{1}$$

for all x in (x_1, x_2). Integrating (1) from x_1 to x_2, one obtains Picone's identity (see [1], p. 226).

The principal application of (1) has been in the proof of a more general form of comparison theorem than that originally given by Sturm. This subject is treated in detail in standard texts on ordinary differential equations, such as [1].

Comparison theorems of the Sturm-Picone type have recently been generalized to elliptic partial differential equations by several authors (see [2] for a list of references) using a variety of techniques. Of interest is the fact that none of these proofs makes use of what appears to be the obvious tool, namely a Picone identity for elliptic equations such as will be described below. While some of these proofs yield results which are somewhat stronger than those which can be derived by this classical method, it does seem of interest to point out how Picone's original argument can be generalized to n-dimensions.

(*) Nella seduta del 19 novembre 1968.

We shall consider functions $u(x)$ and $v(x)$ which are, respectively, non-trivial real solutions of

(2) $$\sum_{i,j=1}^{n} D_j (a_{ij} D_i u) - cu = 0,$$

(3) $$\sum_{i,j=1}^{n} D_j (\alpha_{ij} D_i v) - \gamma v = 0,$$

in a sufficiently smooth, bounded, closed domain $\bar{G} \subset R^n$. The $a_{ij}(x)$ and $\alpha_{ij}(x)$ are to be real and of class C^1 and the matrices (a_{ij}) and (α_{ij}) are to be symmetric and positive definite in $\bar{G}$. The functions $c(x)$ and $\gamma(x)$ are to be real and continuous in $\bar{G}$, and D_i denotes differentiation with respect to the i–th coordinate x_i.

THEOREM 1. *If* $u(x)$ *and* $v(x)$ *are solutions of* (2) *and* (3), *respectively, and* $v(x) \neq 0$ *in* $\bar{G}$, *then*

(4) $$\sum_j D_j \left[\frac{u}{v} \left(v \sum_i a_{ij} D_i u - u \sum_i \alpha_{ij} D_i v \right) \right] =$$

$$= (c - \gamma) u^2 + \sum_{i,j} (a_{ij} - \alpha_{ij}) D_i u D_j u + \sum_{i,j} \alpha_{ij} \left(D_i u - u \frac{D_i v}{v} \right) \left(D_j u - u \frac{D_j v}{v} \right)$$

for all $x \in \bar{G}$.

Proof. The proof is a straightforward expansion of the left side of (4) in which the first term on the right side of (4) is obtained by substituting (2) and (3). The details are left to the reader.

A Sturm-Picone theorem for elliptic equations follows readily from (4).

THEOREM 2. *Suppose* $u(x)$ and $v(x)$ *are solutions of* (2) *and* (3), *respectively, and that* $u(x) = 0$ *on* ∂G. *If*

(*i*) $\Sigma a_{ij} \xi_i \xi_j \geq \Sigma \alpha_{ij} \xi_i \xi_j$ for all real n–tuples $\underline{\xi} = (\xi_1, \cdots, \xi_n)$ and all $x \in G$,

(*ii*) $c(x) \geq \gamma(x)$ in G,

then $v(x)$ *has a zero in* $\bar{G}$.

Proof. Suppose $v(x) \neq 0$ in $\bar{G}$ so that (4) is valid. Integrating the left side of (4) over $\bar{G}$ and applying Green's Theorem, we obtain a boundary integral which vanishes because $u = 0$ on ∂G. Our hypotheses assure that the right side of (4) is non-negative, and therefore the condition

(5) $$\int_G \left[(c - \gamma) u^2 + \sum_{i,j} (a_{ij} - \alpha_{ij}) D_i u D_j u + \right.$$
$$\left. + \sum_{i,j} \alpha_{ij} \left(D_i u - u \frac{D_i v}{v} \right) \left(D_j u - u \frac{D_j v}{v} \right) \right] dx = 0$$

can be satisfied only if the integrand vanishes identically. But then we must have

$$\frac{D_i u}{u} = \frac{D_i v}{v}$$

for $i = 1, \cdots, n$, which assures that u is just a constant multiple of v. Since $u = 0$ on ∂G, this is a contradiction and shows that $v(x) = 0$ for some $x \in \bar{G}$.

As in the one dimensional case this result can be sharpened. It can be shown [3] by means of a modified maximum principle that if ∂G is of bounded curvature and if $v(x)$ satisfies (3) in G and has a zero at some $x_0 \in \partial G$, then the exterior normal derivative cannot vanish at x_0. As in the one dimensional case, this fact allows one to apply the argument of Theorem 2 based only on the assumption that $v(x) \neq 0$ in G. These observations lead to the following stronger result.

COROLLARY 1. *If ∂G is of bounded curvature and the hypotheses of Theorem 2 are satisfied, then either v has a zero in G or else v is a constant multiple of u (and therefore zero everywhere on ∂G).*

One can also use the identity (4) to prove a comparison theorem for the case where $u(x)$ does not vanish everywhere on ∂G. Suppose $u(x)$ and $v(x)$ satisfy

$$\Sigma a_{ij} D_i u + s(x) u = 0, \tag{6}$$

$$\Sigma \alpha_{ij} D_i v + \sigma(x) v = 0, \tag{7}$$

respectively, on ∂G, and we denote the boundary condition $u(x_0) = 0$ by setting $s(x_0) = +\infty$. This notation leads to the following result closely related to Sturm's "second comparison theorem" (see [1], p. 229).

THEOREM 3. *Suppose $u(x)$ satisfies* (2) *and* (6) *and $v(x)$ satisfies* (3) *and* (7) *in* G. *If*

(*i*) $\Sigma a_{ij} \xi_i \xi_j \geq \Sigma \alpha_{ij} \xi_i \xi_j$ for all real n-tuples $\underline{\xi} = (\xi_1, \cdots, \xi_n)$ and all $x \in G$,

(*ii*) $c(x) \geq \gamma(x)$ in G,

(*iii*) $s(x) \geq \sigma(x)$ on ∂G,

then $v(x)$ has a zero in $\bar{G}$ or else v is a constant multiple of u.

Proof. If $v(x) \neq 0$ in $\bar{G}$, then (4) is valid. Substituting (6) and (7) into the left side of (4), integrating over G, and applying Green's theorem, we get

$$\int_{\partial G} (\sigma - s)\, u^2 \, dt$$

which is non-positive by (*iii*). (In case $u(x)$ vanishes on part of ∂G and $s(x) = +\infty$ there, this part of the boundary is omitted in the integration).

Therefore the right side of (5) is at the same time non-positive and non-negative, and therefore is zero. As in Theorem 2, the assumption $v(x) \neq 0$ in $\bar{G}$ leads to the conclusion that v is a constant multiple of u.

This result can again be sharpened in case ∂G is of bounded curvature to yield the following.

COROLLARY 2. *If ∂G is of bounded curvature and the hypotheses of Theorem 3 are satisfied, then either v has a zero in G or else v is a constant multiple of u.*

As an application of Corollary 1, we shall give a simple proof of the fact that for domains of bounded curvature, the smallest eigenvalue of the elliptic eigenvalue problem

$$(8) \qquad Lu \equiv -\Sigma D_j (a_{ij} D_i u) + cu = \lambda u \quad \text{in} \quad G$$
$$u = 0 \quad \text{on} \quad \partial G,$$

is a strictly decreasing function of the domain (1). Our only recourse to the variational theory of eigenvalues [4] is the fact that the first eigenvalue of (8) is simple and that the corresponding eigenfunction does not vanish in G.

THEOREM 4. *Let G and G′ be domains of bounded curvature satisfying $G \subsetneqq G'$. If λ_1 and λ'_1 are the first eigenvalues of (8) in G and G′ respectively, then $\lambda_1 > \lambda'_1$.*

Proof. By hypothesis, the boundary value problems

$$(9) \qquad Lu = \lambda_1 u \quad \text{in} \quad G$$
$$u = 0 \quad \text{on} \quad \partial G$$

and

$$(10) \qquad Lv = \lambda'_1 v \quad \text{in} \quad G'$$
$$v = 0 \quad \text{on} \quad \partial G'$$

have solutions u and v which are positive in G and G′, respectively. If $\lambda_1 \leq \lambda'_1$, then Corollary 1 can be applied to (9) and (10) to conclude that $v(x)$ has a zero in G or else is a constant multiple of u. Both of these conclusions contradict the positivity of $v(x)$ in G′.

REFERENCES.

[1] E. L. INCE, *Ordinary Differential Equations*, Dover 1956.

[2] K. KREITH, *Sturmian theorems and positive resolvents*, « Trans. Amer. Math. Soc. », to appear.

[3] K. KREITH, *A remark on a comparison theorem of Swanson*, « Proc. Amer. Math. Soc. », to appear.

[4] R. COURANT and D. HILBERT, *Methods of Mathematical Physics*, « Interscience » (1953).

(1) The fact that the eigenvalues of (8) are monotonically decreasing functions of G follows easily from the variational theory for eigenvalues (see [4], p. 409). The fact that they are strictly decreasing requires additional analysis.