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Partially hypo-analytic distributions and pseudo-differential operators

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Analisi matematica (Equazioni differenziali). — *Partially hypoanalytic distributions and pseudo-differential operators.* Nota di VIOREL BARBU, presentata (*) dal Socio M. PICONE.

RIASSUNTO. — In questa Nota si stabilisce un risultato concernente la regolarità degli operatori pseudo-differenziali.

Pseudo-differential operators and their local properties, have been developed by Kohn-Nirenberg [7], Hörmander [6], Volevič [8] and [9], L. Boutet de Monvel et P. Krée [1], Zaidman [10]. In connection with a recent paper by Volevič [9], we study in the present Note, the partially hypoanalyticity of some classes of pseudo-differential operators.

1. NOTATIONS.

We set $D_j = -i\partial/\partial x_j$, $\partial_j = \partial/\partial\xi_j$ for $1 \leq j \leq n$, and for each n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$, we set $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$; $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$; $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$; $|\alpha| = \sum_{j=1}^{n} \alpha_j$. By S we denote the space of C^∞ complex valued functions $\varphi(x)$, such that $\sup_{x \in R^n} |x^\beta D^\alpha \varphi(x)| < \infty$ for all multi-indices α and β. Let $n = n' + n''$. If we consider the space R^n as a product $R^n = R^{n'} \times R^{n''}$, then the general point of R^n will be denoted by $x = (x', x'')$; where $x' \in R^{n'}$ and $x'' \in R^{n''}$.

For real t and s, we introduce the norm

$$\|u\|_{s,t}^2 = (2\pi)^{-n} \int |\hat{u}(\xi)|^2 (1 + |\xi'|^2)^{s/2} (1 + |\xi''|^2)^{t/2} d\xi, \quad u \in S \tag{1.1}$$

where u is the Fourier transform of u. Let $H^{s,t}$ be the space obtained by the completion of S in this norm. We set

$$H^{-\infty,t} = \bigcup_{s=-\infty}^{\infty} H^{s,t} \quad , \quad H^{-\infty} = \bigcup_{-\infty}^{\infty} H^{s,s} .$$

If K is any compact set of R^n, we shall use the notations

$$\|u, K\| = \left(\int_K |u(x)|^2 dx\right)^{1/2}, \|u, K\|_\infty = \text{ess}_K \sup |u(x)| .$$

A C^∞ function $u(x)$ defined in an open subset $\Omega \subset R^n$ is said to be hypoanalytic of class ρ, $1 \leq \rho \leq \infty$, if for any compact set $K \subset \Omega$ there

(*) Nella seduta del 19 novembre 1968.

exists a constant M such that for any multi-index, the following inequality be true

$$(1.2) \qquad \| D^{\alpha} u , K \|_{\infty} \leq M^{|\alpha|+1} \Gamma (\rho \, | \alpha |),$$

where Γ is Euler's function. The Gevrey class $G^{\varrho}(\Omega)$ is the space of all functions that are of class ρ on Ω. If $\rho > 1$, $G_0^{\varrho}(\Omega)$ will denote the space $C_0^{\infty}(\Omega) \cap G^{\varrho}(\Omega)$.

Let $u \in D'(\Omega)$ and $\Omega' \subset R^{n'}$, $\Omega'' \subset R^{n''}$ be two open sets such that $\Omega' \times \Omega'' \subset \Omega$. If $\varphi \in C_0^{\infty}(\Omega'')$, then we consider the distribution $u_{\varphi} \in D'(\Omega')$ defined by the relation

$$(1.3) \qquad u_{\varphi}(\psi) = u(\varphi\psi) \quad , \quad \forall \psi \in C_0^{\infty}(\Omega').$$

The distribution $u \in D'(\Omega)$ is said to be ρ–partially hypoanalytic on Ω if $u_{\varphi} \in G^{\varrho}(\Omega')$ whenever $\varphi \in C_0^{\infty}(\Omega')$ and $\Omega' \times \Omega'' \subset \Omega$. In following, for simplicity, we denote by $G_{x'}^{\varrho}(\Omega)$ the space of all distributions ρ–partially hypoanalytic with respect to x', in open set $\Omega \subset R^n$.

Pseudo-differential operators. Pseudo-local properties.

We consider pseudo-differential operators of the form

$$(1.4) \qquad Au(x) = (2\pi)^{-n} \int e^{i\langle x,\xi \rangle} a(x,\xi) \hat{u}(\xi) \, d\xi \quad , \quad u \in S,$$

with the symbol $a(x,\xi) = a(\xi) + a'(x,\xi)$, where $a'(x,\xi)$ as a function of x, vanishes at ∞.

Concerning the symbol $a(x,\xi) \in C^{\infty}(R^n \times R^n)$, we shall assume that there are constants m, M, M_1, independent of α and an increasing function N(r), such that

$$\text{I)} \qquad | \partial^{\alpha} a(\xi) | \leq M_1^{|\alpha|} (1 + |\xi'|)^{m - \frac{|\alpha'|}{\varrho}} (1 + |\xi''|)^{N(|\alpha''|)}$$

$$\text{II)} \qquad \int | D^{\beta} \partial^{\alpha} a'(x,\xi) | \, dx \leq M^{|\alpha|+|\beta|+1} \Gamma(\rho \, |\beta|) (1 + |\xi'|)^{m - \frac{|\alpha'|}{\varrho}} (1 + |\xi''|)^{N(|\alpha''|)} .$$

It is easy to prove that, under conditions I), II), the operator A can be extended by continuity to a continuous linear map of $H^{-\infty}$ into itself. Let Ω be an open set of R^n and let $\tilde{K} \in D'(R^n \times R^n)$ be a distribution defined by

$$(1.5) \qquad \tilde{K}(F) = \int e^{i\langle x,\xi \rangle} a(x,\xi) (1 + |\xi''|^2)^{-l} \hat{F}(x,\xi) \, d\xi \, dx \, ; \; F \in C_0^{\infty}(R^n \times R^n)$$

where $\hat{F}(x,\xi) = \int e^{-i\langle x,\xi \rangle} F(x,y) \, dy$ and $2l \geq N(0) + n'' + 1$.

If $K \in D'(R^n \times R^n)$ is the kernel of pseudo-differential operator A, i.e. $(Au, v) = K(u \otimes v)$, $\forall u, v \in S$, then we have the equality

$$(1.6) \qquad (1 - \Delta_{y''})^l \tilde{K} = K,$$

where $\Delta'' = \sum_{j=n'+1}^{n} D_j^2$.

LEMMA 1. *The distribution* $\tilde{K}$ *is a continuous function and* ρ*-hypoanalytic with respect to* x', *outside the diagonal* $\{(x, y) \in R^n \times R^n ; x' = y'\}$.

Proof. Let us remark that

$$(1.7) \qquad (x' - y')^q D_x^p, \tilde{K}(F) = \sum_{|\delta| \le |p|} (-1)^{p+\delta} \binom{\delta}{p} \int e^{i\langle x,\xi\rangle} (1 + |\xi''|^2)^{-l} \hat{F}(x, \xi) \partial_{\xi'}^q ((\xi')^\delta D_{x'}^{p-\delta} a(x, \xi)) \, dx \, d\xi$$

for any multi-indices p and q. Let $f_{q,p}$ be the function, $f_{q,p}(x, \xi) = (-1)^{q+p} e^{i\langle x,\xi\rangle} \sum_{|\delta| \le |p|} \binom{\delta}{p} (1 + |\xi''|^2)^{-l} D_{\xi'}^q ((\xi')^\delta D_{x'}^{p-q} a(x, \xi))$. If we choose $|q|$ so large that

$$\rho(|p| + m + n' + 1) \le |q| \le \rho(|p| + m + n' + 1) + 1,$$

it follows that

$$(1.8) \qquad (x' - y')^q D_{x'}^p \tilde{K}(x, y) = \int e^{-i\langle y,\xi\rangle} f_{p,q}(x, \xi) \, d\xi.$$

On the other hand, using the conditions I), II), we obtain

$$(1.9) \qquad |f_{p,q}(x, \xi)|_\infty \le M^{|p|+1} \Gamma(\rho |p|) (1 + |\xi'|)^{-1} (1 + |\xi''|)^{-n''+1}$$

with $|q|$ defined above. Consequently

$$\sup_{|x'-y'| \ge \varepsilon} |D_{X'}^p \tilde{K}(x, y)| \le M_1^{|p|+1} \Gamma(\rho |p|).$$

Remark. Applying Parseval's formula, from (1.8), (1.9) we obtain

$$(1.10) \qquad \int |(1 - \zeta(x' - y'))|^2 |D_{x'}^p \tilde{K}(x, y)|^2 \, dy \le M^{|p|+1} \Gamma(\rho |p|),$$

where $\zeta(x')$ is a C_0^∞ function equal to 1 on some neighborhood of the origin.

Let Ω be an open set of R^n and l an integer such that $2l \ge N(0) + n'' + 1$.

THEOREM 1. *Let* $a(x, \xi)$ *be a* C_0^∞ *symbol satisfying I), II) and let A be its pseudo-differential operator. If* $u \in H^{-\infty,l} \cap G_{x'}^\rho(\Omega)$ *then* $Au \in G_{x'}^\rho(\Omega)$, $(\rho \ge 1)$.

The following lemma is a sharp form of a result due to Friberg [4].

LEMMA 2. *Let* $u \in \varepsilon'(R^n) \cap H^{-\infty,l}$ *and* $\rho > 1$. *In order that* $u \in G_{x'}^\rho$, *it is necessary and sufficient that there exist* $M, A > 0$, *such that*

$$(1.11) \qquad |\hat{u}(\xi)| \le M(1 + |\xi''|)^{-l} \exp(-A|\xi'|^{1/\rho}), \qquad \xi \in R^n.$$

Proof of Theorem 1. First we assume that $\rho > 1$. Let $\Omega' \subset R^{n'}$, $\Omega'' \subset R^{n''}$ be two bounded sets such that $\Omega' \times \Omega'' \subset \Omega$. We set $D = \Omega' \times \Omega''$ and construct $\varphi_1 \in C_0^\infty(R^{n'})$, $\varphi_2 \in C_0^\infty(R^{n''})$ so that $\varphi_1 = 1$ on Ω', $\varphi_2 = 1$ on Ω'', supp $\varphi_1 \otimes \varphi_2 \subset \Omega$. Let us put $g = \varphi_1 \otimes \varphi_2$. If $\psi \in C_0^\infty(\Omega'')$, from (1.4) we have

$$(1.12) \qquad |D^\alpha A(gu)_\psi(x')| \leq (2\pi)^{-n} \sum_{|\beta<|\alpha|} \binom{\beta}{\alpha} \int |\psi(x'')| \, |D^{\alpha-\beta} a(x,\xi)| \, |\widehat{D_{\xi'}^\beta (ug)}| \, d\xi \, dx'' .$$

Since $u \in G_{x''}^\varrho$, using Lemma 2, we obtain

$$(1.13) \qquad \| D^\alpha A(gu)_\psi \|_\infty \leq M^{|\alpha|+1} \Gamma(\rho |\alpha|) .$$

We set $u^* = ug$ and choose $\varphi \in C_0^\infty(\Omega')$ equal to 1 in an arbitrary compact set of Ω'. For $u \in S(R^n)$, $v \in S(R^{n'})$ and $\psi \in C_0^\infty(\Omega'')$ we have

$$(1.14) \qquad ((1-\Delta')^k \varphi(Au^*)_\psi , v) = \tilde{K}((1-\Delta'')^l u^* \otimes \varphi\psi (1-\Delta')^k v) .$$

Then from Lemma 1 it follows

$$(1.15) \qquad ((1-\Delta')^k \varphi(Au)_\psi , v) = \int A_k(x,y)(1-\Delta'')^l u(y)\psi(x'') v(x') \, dx \, dy$$

where $A_k(x,y) = (1-\zeta(x'-y'))(1-\Delta')_{x'}^k (\tilde{K}(x,y)\varphi(x'))$. Here $\zeta(x')$ is a C_0^∞ function equal to 1 in a sufficiently small neighborhood of the origin. From the remark 1 we obtain

$$(1.16) \quad \|(1-\Delta')^k \varphi(Au^*)_\psi\| \leq \|u^*\|_{0,l} \sum_{|\beta|<2k} M^{2k-|\beta|+1} \Gamma((2k-|\beta|)\rho) \|D^\beta \varphi\| .$$

Since the inequality (1.16) is true for every $u \in H^{-\infty,l}$, choosing $\varphi \in G_0^\varrho(\Omega')$ it follows $A((1-g)u) \in G_{x'}^\varrho(\Omega)$. Because the hypoanalyticity is a local property from (1.13) it follows $u \in G_{x'}^\varrho(\Omega)$. If $\rho = 1$, a same conclusion follows taking in (1.16) $\varphi = \varphi_{2k}$ when $\varphi_k(x')$ is a sequence of $C_0^\infty(\Omega')$ functions such that $\varphi_k(x') = 1$ in an arbitrary compact set of Ω' and satisfies

$$(1.18) \qquad \| D^\alpha \varphi_k \|_\infty \leq C^{k+1} k^{|\alpha|} , \quad \text{for } |\alpha| \leq k .$$

2. PARTIALLY HYPOELLIPTIC PSEUDO-DIFFERENTIAL OPERATORS.

Let $a(x,\xi)$ be the symbol considered above. Assume that there are some nonnegative constants N, N_1, N_2, M such that

$$a) \quad |\partial^\alpha a(\xi)| \leq M(1+|\xi'|)^{m-\frac{|\alpha|}{\varrho}} (1+|\xi''|)^N ;$$

$$b) \quad \int |D^\beta \partial^\alpha a'(x,\xi)| \, dx \leq M_\alpha M^{|\alpha|+1} \Gamma(\rho|\beta|)(1+|\xi'|)^{m-\frac{|\alpha|}{\varrho}} (1+|\xi''|)^N ;$$

where $M_\alpha \leq M^{|\alpha|+1}$ if $\rho \geq 2$ and $M_\alpha \leq M^{|\alpha|+1}/\Gamma\left(\frac{4|\alpha|}{\rho}\right)$, if $1 < \rho < 2$;

$$c) \quad |a(\xi) - a(\eta)| \leq M (1 + |\xi - \eta|)^{N_1} (1 + |\eta'|)^{m-\sigma} (1 + |\eta''|)^{N};$$

$$d) \quad \int |D^{\beta}_{x'} (a'(x, \xi) - a'(x, \eta))| \, dx \leq M^{|\beta|+1} \Gamma(\rho |\beta|) (1 + |\xi - \eta|)^{N_2}$$

$$(1 + |\eta'|)^{m-\sigma} (1 + |\eta''|)^{N}.$$

with a real $\sigma \geq 1$;

$$e) \quad |a(\xi)|, |a'(x, \xi)| \geq c (1 + |\xi|)^{m} (1 + |\xi''|)^{N}$$

for $|\xi'|$ sufficiently large.

THEOREM 2. *Let $a(x, \xi)$ be a C^∞ symbol which satisfies a), b), c), d), e) and let A its associated pseudo-differential operator. If $u \in H^{-\infty, \frac{N+n''+1}{2}} \cap G^{\varrho}_{x''}(\Omega)$ is a solution of the equation $Au = f$ with $f \in G^{\varrho}(\Omega)$, $\rho > 1$, then $u \in G^{\varrho}(\Omega)$, (A is conditionnally ρ-hypoelliptic pseudo-differential operator).*

We consider some results which are required in the proof of theorem 2. Call $a_{\alpha\beta}(x, \xi) = D^{\alpha-\beta} a(x, \xi)$ and denote by $A_{\alpha\beta}$ the pseudo-differential operator associated to it. If $\varphi(x)$ is a C_0^∞ function, $[A_{\alpha\beta}, \varphi]$ will denote the commutator of $A_{\alpha\beta}$ with φ. We assume that the condition *e*) is verified for $|\xi'| \geq R$, and let $\chi(\xi')$ be a C^∞ non-negative function equal one for $|\xi'| > R + 1$ and vanishes for $|\xi'| \leq R$. Let E and G be pseudo-differential operators with the symbols $e(x, \xi) = \chi(\xi')/a(x, \xi)$ and $\chi(\xi')$. We denote by T the operator $EA - G$. By a well known argument (see Kohn-Nirenberg [7], Volevič [9]) it follows

LEMMA 3. *If t, s are real numbers, there exists a non-negative constant $C_{t,s}$ such that*

$$\|Eu\|_{s,t} \leq C_{t,s} \|u\|_{s-m, t-N}; \qquad u \in S, \tag{2.2}$$

$$\|Tu\|_{s,t} \leq C_{t,s} \|u\|_{s-\sigma, t}; \qquad u \in S \tag{2.3}$$

and

$$\|[A_{\alpha\beta}, \varphi] u\|_{s,t} \leq C_{t,s} M^{|\alpha-\beta|+1} \Gamma(\rho |\alpha - \beta|) \|u\|_{s+m-1, t+N}. \tag{2.4}$$

LEMMA 4. *Let A be a pseudo-differential operator satisfying the conditions a), b), c), d), e) and $u \in H^{-\infty} \cap G^{d}_{x''}(\Omega)$ with $d > 1$. If $Au \in C^\infty(\Omega)$ then $u \in C^\infty(\Omega)$.*

Proof. We see an argument of Volevič [8]. Let $\varphi(x) \in C_0^\infty(\Omega)$ such that $\varphi(x) = 1$ on $\Omega' \subset \Omega$ and let ψ be a C_0^∞ function equal to 1 in the support of φ. If $T_1 = I - G$, then we may write φu in the form

$$\varphi u = EA(\varphi u) - T(\varphi u) + T_1(\varphi u). \tag{2.5}$$

Since $u \in G^{d}_{x''}$, it follows (see Lemma 2)

$$|\hat{\varphi u}(\xi)| \leq C (1 + |\xi'|)^{k} \exp(-|\xi''|^{1/d}).$$

Hence $\varphi u \in H^{k,\infty}$ with a fixed k. Now, we remark that $A(\varphi u) = \varphi A u + [A, \varphi](\psi u)$. This implies that $A(\varphi u) \in H^{m+k-1,\infty}$. Using lemma 3 it follows from (2.5) that $\varphi u \in H^{k-1,\infty}$. By iteration we deduce that $\varphi u \in H^{\infty}$, hence $u \in C^{\infty}(\Omega)$.

Proof of Theorem 2. Let $u \in G^{\varrho}_{x''}(\Omega) \cap H^{-\infty, \frac{N+n''+1}{2}}$ and $Au \in G^{\varrho}(\Omega)$. By lemma 4 we may suppose that $u \in C^{\infty}(\Omega)$. It is enough to prove that every point in Ω has an open neighborhood ω in which $u \in G^{\varrho}$. In the following, ω_{ε} denotes the set of the points of ω at distance $> \varepsilon$ from $C\omega$. Let φ, ψ be two C_0^{∞} functions so that supp $\varphi \subset \omega_{2\varepsilon}$, supp $\psi \subset \omega_{\varepsilon}$, $\varphi(x) = 1$ on $\omega_{3\varepsilon}$ and $\psi(x) = 1$ on $\omega_{2\varepsilon}$. If we denote $u_1 = u\psi$, then by an elementary calculus it follows

$$\varphi D^{\alpha} u_1 = E\varphi D^{\alpha} A u_1 - \sum_{|\beta|<|\alpha|} EA_{\alpha\beta}(\varphi D^{\beta} u_1) - T(\varphi D^{\alpha} u_1) + \sum_{|\beta|\le|\alpha|} \binom{\beta}{\alpha} E[A_{\alpha\beta}, \varphi] D^{\beta} u_1 + T_1(\varphi D^{\alpha} u_1). \tag{2.6}$$

We have the estimate

$$\|\varphi D^{\alpha} A(1-\psi) u\| \le M\, \Gamma^{|\alpha|+1}(\rho|\alpha|). \tag{2.7}$$

For $\alpha = \alpha'$ this formula is a consequence of theorem 1 and for $\alpha = \alpha''$ it follows using the fact that $u \in G^{\varrho}_{x''}(\Omega)$. Applying lemma 2, from (2.6) we deduce the inequality

$$\begin{aligned}\|D^{\alpha} u, \omega_{3\varepsilon}\| \le &\sum_{|\beta|<|\alpha|} M^{|\alpha-\beta|+1} \Gamma(\rho|\alpha-\beta|) \|D^{\beta} u; \omega_{2\varepsilon}\| + \\ &+ \sum_{|\beta|\le|\alpha|-1} \binom{\beta}{\alpha} \|D^{\beta} u; \omega_{\varepsilon}\| (\|D^{\alpha-\beta}\psi\| + \|D^{\alpha-\beta}\varphi\|) + \\ &+ \sum_{|\beta|\le|\alpha|-1} M^{|\alpha-\beta|+1} \Gamma(\rho|\alpha-\beta|) \sum_{|\gamma|\le|\beta|} \binom{\gamma}{\beta} \|D^{\beta-\gamma} u; \omega_{\varepsilon}\| \|D^{\gamma}\varphi\|.\end{aligned} \tag{2.8}$$

To prove the theorem, we first assume that $\rho \ge 2$. We choose $\varphi_k \in C_0^{\infty}(\omega_{(k-1)\varepsilon})$, $\psi_k \in C_0^{\infty}(\omega_{(k-2)\varepsilon})$ such that $\varphi_k = 1$ on $\omega_{k\varepsilon}$, $\psi_k = 1$ on $\omega_{(k-1)\varepsilon}$ and

$$\|D^{\alpha}\varphi_k\|_{\infty} \le M_1^{k+1} k^{|\alpha|} \qquad \text{for} \quad |\alpha| \le k; \tag{2.9}$$

$$\|D^{\alpha}\psi_k\|_{\infty} \le M_1^{k+1} k^{|\alpha|} \qquad \text{for} \quad |\alpha| \le k. \tag{2.10}$$

If in (2.6) we take $\varphi = \varphi_k$ and $\psi = \psi_k$, we obtain from (2.8)

$$\begin{aligned}\|D^{\alpha} u; \omega_{|\alpha|\varepsilon}\| \le &\sum_{|\beta|<|\alpha|} M^{|\alpha-\beta|+1} \Gamma(\rho|\alpha-\beta|) \|D^{\beta} u; \omega_{(|\alpha|-1)\varepsilon}\| + \\ &+ \sum_{|\beta|\le|\alpha|} M^{|\alpha-\beta|+1} \Gamma(\rho|\alpha-\beta|) \|D^{\beta} u; \omega_{(|\alpha|-2)\varepsilon}\|.\end{aligned} \tag{2.11}$$

Take ε such that $|\alpha|\varepsilon \le c$, where c is a non-negative constant sufficiently small. Then by recurrence on $|\alpha|$ we obtain the desired inequality

$$\|D^{\alpha} u; \omega_c\| \le M^{|\alpha|+1} \Gamma(\rho|\alpha|). \tag{2.12}$$

Now we assume that $1 < \rho < 2$. Let $\chi_k(x)$ be a sequence of C_0^∞ functions such that $\chi_k(x) = 1$ on $\omega_{(k-1)\varepsilon}$ and satisfies: $\|D^\alpha \chi_k\|_\infty \leq C^{k+1} k^{|\alpha|}$ for $|\alpha| \leq k$.

To estimate the L_2–norm of $[A_{\alpha\beta}, \varphi_k] D^\beta(u\chi_k)$, we write it in the form

$$[A_{\alpha\beta}, \varphi_k] D^\beta(u\chi_k) = [A_{\alpha\beta}, \varphi_k] \psi_k D^\beta(u\chi_k) + [A_{\alpha\beta}, \varphi_k](1 - \psi_k) D^\beta(\chi_k u). \tag{2.13}$$

Since $1 - \psi_k$ and φ_k have disjoint support using the conditions *a*), *b*) it follows as in [2]

$$\|[A_{\alpha\beta}, \varphi_{|\alpha|}](1 - \chi_{|\alpha|}) D^\beta u_1\|_{-m} \leq M^{|\alpha-\beta|+1} \Gamma(\rho|\alpha - \beta|) / \tag{2.14}$$

$$/ \Gamma(N(1 + 4/\rho) \|(1 - \chi_{|\alpha|}) \psi_{|\alpha|} D^\beta u\|_{-\sigma} \|\varphi_{|\alpha|}\| N(1 + 1/\rho)$$

where $\rho(\sigma + n + 1) \leq N \leq \rho(\sigma + n + 1) + 1$. Then from (2.6) it follows

$$\|D^\alpha u_1 ; \omega_c\| \leq M^{|\alpha|+1} \Gamma(\rho|\alpha|). \tag{2.15}$$

Hence $u \in G^\varrho(\Omega)$ and the proof is completed.

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