Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Viorel Barbu

On regularity of weak solutions of abstract differential equations in Hilbert spaces

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **45** (1968), n.3-4, p. 129–134.

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<http://www.bdim.eu/item?id=RLINA_1968_8_45_3-4_129_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1968.

Equazioni differenziali. — *On regularity of weak solutions of abstract differential equations in Hilbert spaces.* Nota (*) di VIOREL BARBU, presentata dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota si stabilisce un risultato concernente la regolarità delle soluzioni deboli delle equazioni differenziali negli spazi di Hilbert.

Let H be a Hilbert space; $\langle\ \rangle$ and $\|\ \|$ are the notations for the scalar product and for the norm. If Δ is an open interval of the real axis R, we denote by $D(\Delta, H)$ the space of infinitely differentiable H–valued functions with compact support in Δ. We consider on $D(\Delta, H)$ the usual topology and denote by $D'(\Delta, H)$ its dual (see L. Schwartz [6]). If $u \in D'(\Delta, H)$ then for each compact set $K \subset \Delta$ there exists a non-negative integer 1 such that

$$(1) \qquad |u(\varphi)| \leq C_K \sum_{j=0}^{l_K} \sup \| D^j \varphi(t) \| \quad , \quad \varphi \in D(\Delta, H) \ \operatorname{supp} \varphi \subset K .$$

If there exists an entire l such that $l_K \leq l$ for any $K \subset \Delta$, then the number 1 is called the order of distribution u on Δ. In particular the distribution defined by $f(\varphi) = \int \langle f(t), \varphi(t) \rangle \, dt$, with $f \in L^1_{loc}(R, H)$, is of order zero on the real axis R.

Let $\{A(t)\}$ be a family of closed, linear and dense operators in H and let $\{A^*(t)\}$ be the family of their adjoints. We assume that the domain D_{A^*} of $A^*(t)$ is independent of t. Denote by $D(\Delta, D_{A^*})$ the space of functions $\varphi(t)$ from Δ into D_{A^*} such that $A^*(t)\varphi(t), \varphi(t) \in D(\Delta, H)$. Since $\overline{D_{A^*}} = H$ it is easily to show that the space $D(\Delta, D_A)$ is dense in $D(\Delta, H)$.

Let $L^* : D(R, D_{A^*}) \to D(R, H)$ be the linear operator

$$(2) \qquad L^* = -\left(\frac{1}{i} \frac{d}{dt} + A^*(t)\right) \cdot$$

Definition. We say that the distribution $u \in D'(\Delta, H)$ is a weak solution on Δ of the equation

$$(E) \qquad \frac{1}{i} \frac{du}{dt} + A(t) u = f$$

where $f \in D'(\Delta, H)$ if the following relation

$$(3) \qquad (u, L\varphi) = (f, \varphi)$$

holds for any $\varphi \in D(\Delta, D_{A^*})$.

The regularity theorem for the strict solutions of (E) have been obtained by T. Kato and H. Tanabe [5], H. Tanabe [7], S. Agmon and L. Nirenberg [1].

(*) Pervenuta all'Accademia il 28 agosto 1968.

When $A(t) = A$ is independent of t, the writer has given in [2] some conditions which ensure the differentiability and hypoanalyticity of the weak solutions of the equation (E). In this paper we extend these results for variable operators $A(t)$.

Assumptions. (i). For each $t \in [-a, a]$, $A^*(t)$ is a densely defined linear closed operator in H and the domain $D_{A^*} = D(A^*(t))$ is independent of t. For any $x \in D_{A^*}$ the function $t \to A(t)x$ is infinitely differentiable.

(ii). There exists a sequence of positive constants $\{C_m\}_{m=0}^{\infty}$ such that $(\lambda I - A^*(t))^{-1}$ exists in the domain

$$\Sigma_m = \{\lambda ; |Im \lambda| \leq m \log |\lambda| ; |\lambda| \geq C_m\}$$

(iii). There exist constants $M, N, d > 0$ such that for all $|t| < a$ and non-negative integers k,

$$\|(\partial/\partial t)^k (\sigma I - A^*(t))^{-1}\| \leq B_k |\sigma|^{-d} \qquad \text{for} \quad \sigma \in R, \ |\sigma| \geq C_0 \tag{4}$$

$$\|(\partial/\partial t)^k (\lambda I - A^*(t))^{-1}\| \leq B_{km} |\lambda|^M \exp(N |Im \lambda|) \qquad \text{for} \quad \lambda \in \Sigma_m \tag{5}$$

in the uniform operator topology. Here B_k, B_{km} are positive constants.

THEOREM. *Let* $u \in D'((-a, a), H)$ *be a weak solution of the equation* (E) *in the interval* $(-a, a)$ *and let* l *be its order on this set. Suppose that* $f \in C^{k+1}((-a, a), H)$, k *being some positive integer. Let* $a' = a - d^{-1}N(k + l + 1/2) - N$ *and assume that* $a' > 0$. *Under assumptions* (i)–(iii), $u \in C^{k-1}$ *in the interval* $|t| < a'$.

COROLLARY. Let $u \in D'(R, H)$ be a weak solution in R of the equation (E). If $f \in C^{\infty}(R, H)$ and the assumptions (i)–(iii) are satisfied then $u \in C^{\infty}(R, H)$.

Proof of Theorem. In order to prove the differentiability of u we will first construct a "parametrix" associated to operator L^* (see L. Hormander [3]). For $\lambda \in \Sigma_m$ denote by $E_0(t, \lambda)$ the operator

$$E_0(t, \lambda) = (\lambda I + A^*(t))^{-1} \qquad |t| < a$$

and define successively operators $E_j(t, \lambda)$ by means of the recursion formula

$$E_{j+1}(t, \lambda) + (\lambda I + A^*(t))^{-1} D_t^1 E_j(t, \lambda) = 0 \qquad \lambda \in \Sigma_m \tag{6}$$

where $D_t^1 = 1/i \, \partial/\partial t$. It follows from (4) and (5) that

$$\|D_t^k E_j(t, \sigma)\| \leq B_{jk} |\sigma|^{-d(j+1)}, \qquad \sigma \in R, \ |\sigma| \geq C_0 \tag{7}$$

$$\|D_t^k E_j(t, \lambda)\| \leq B_{jk}^1 |\lambda|^{M(j+1)} \exp(N(j+1) |Im \lambda|), \qquad \lambda \in \Sigma_m \tag{8}$$

Putting

$$E(t, \lambda) = \sum_{j=0}^{n} E_j(t, \lambda) \qquad |t| < a, \ \lambda \in \Sigma_m \tag{9}$$

we get

$$\text{(10)} \qquad L^*(e^{it\lambda} E(t,\lambda)) = -e^{it\lambda}(I + D_t^1 E_n(t,\lambda)).$$

Let $\varphi \in D((-a, a), D_{A^*})$. Using the fact that the operator A^* is closed we deduce that $E(t,\lambda)\hat{\varphi}(\lambda) \in D_{A^*}$ for $|t| < a$ and $\lambda \in \Sigma_m$. Here

$$\hat{\varphi}(\lambda) = \int e^{-it\lambda} \varphi(t)\, dt$$

is the Fourier-Laplace transform of $\varphi(t)$. Operating under the integral sign we find that (10) is equivalent to

$$\text{(11)} \quad \varphi(t) = -(2\pi)^{-1} L^*\left(\int_{|\sigma| \geq C_m} e^{it\sigma} E(t,\sigma)\hat{\varphi}(\sigma)\, d\sigma\right) + (2\pi)^{-1}\int_{|\sigma| \leq C_m} e^{it\sigma}\hat{\varphi}(\sigma)\, d\sigma -$$

$$-(2\pi i)^{-1}\left(\int_{|\sigma| \geq C_m} e^{it\sigma} D_t^1 E_n(t,\sigma)\hat{\varphi}(\sigma)\, d\sigma\right).$$

Let $\rho(t)$ be a C_0^∞ scalar function such that supp $\rho \subset \{t; |t| < a\}$ and $\rho(t) = 1$ for $|t| < a - \varepsilon'$ where $0 < \varepsilon' < \varepsilon$ and $\varepsilon > d^{-1} N(k + 1 + 1/2) + N$. Denote by Δ the open interval $\{t; |t| < a - \varepsilon\}$. Obviously for $u \in D'(\Delta, H)$ we have

$$u(\varphi) = \rho u(\varphi) \quad , \quad \varphi \in D(\Delta, H).$$

If $u \in D'((-a, a), H)$ is a weak solution of (E) on $(-a, a)$ then from (11) it follows

$$(-1)^j D^j u(\varphi) = -(2\pi)^{-1}\int\int_{|\sigma| \geq C_m} e^{it\sigma}\langle \rho(t) f(t), E(t,\sigma)\widehat{D^j\varphi}(\sigma)\rangle\, d\sigma\, dt +$$

$$+(2\pi)^{-1} u\left(\rho(t)\int_{|\sigma| \leq C_m} e^{it\sigma}\widehat{D^j\varphi}(\sigma)\, d\sigma\right) - (2\pi i)^{-1} u\left(\rho(t)\int_{|\sigma| \geq C_m} e^{it\sigma} D_t^1 E_n(t,\sigma)\widehat{D^j\varphi}(\sigma)\, d\sigma\right) +$$

$$+ u\left(D_t^1 \rho(t)\int_{|\sigma| \geq C_m} e^{it\sigma} E(t,\sigma)\widehat{D^j\varphi}(\sigma)\, d\sigma\right)$$

for any $\varphi \in D(\Delta, D_{A^*})$ and for any non-negative integer j. Here $D^j u$ is the derivative in the sense of distributions of u. Let $E^*(t,\lambda)$ be the adjoint of the operator $E(t,\lambda)$. As is easily seen, we have

$$\text{(12)} \qquad \int\int_{|\sigma| \geq C_m} e^{it\sigma}\langle \rho(t) f(t), E(t,\sigma) D^j\varphi(\sigma)\rangle\, d\sigma\, dt =$$

$$= \int_{|\sigma| \geq C_m} \sigma^{-1}\left\langle \int e^{it\sigma} D_t^{j+1}(E^*(t,\sigma)\rho(t) f(t))\, dt, \hat{\varphi}(\sigma)\right\rangle d\sigma.$$

The right term in (12) is dominated by $M_j |\varphi|_{L^2}$ for $0 \le j \le k$. Here we used the fact that $f \in C^{k+1}$ and easily verifiable inequality

$$\| D_t^{j+1} (E^*(t, \sigma)\ \rho(t) f(t)) \| \le B_j |\sigma|^{-d}, \qquad 0 \le j \le k.$$

Hence

$$(13) \qquad \left| \int \int_{|\sigma| \ge C_m} e^{it\sigma} \langle \rho(t) f(t), E(t,\sigma) \widehat{D^j \varphi}(\sigma) \rangle \, d\sigma \, dt \right| \le M_j \|\varphi\|_{L^2}, \qquad 0 \le j \le k.$$

for any $\varphi \in D(\Delta, D_{A^*})$. Since supp $\rho \subset (-a, a)$, from (1) we have

$$(14) \qquad \left| u\left(\rho(t) \int_{|\sigma| \ge C_m} e^{it\sigma} D_t^1 E_n(t,\sigma) \widehat{D^j \varphi}(\sigma) \, d\sigma \right) \right| \le$$

$$\le C \sum_{p=0}^{l} \sup \| D^p (\rho(t)\, e^{it\sigma} D_t^1 E_n(t,\sigma)) \, |\sigma|^j \, \|\hat{\varphi}(\sigma)\| \, d\sigma.$$

Expanding $D^p(\rho(t)\, e^{it\sigma} D_t^1 E_n(t,\sigma))$ by Leibniz's formula and applying the Schwartz's inequality we obtain

$$\left| u\left(\rho(t) \int_{|\sigma| \ge C_m} e^{it\sigma} D^j \varphi(\sigma) D_t^1 E_n \, d\sigma \right) \right| \le C_j \|\varphi\|_{L^2} \left(\int |\sigma|^{2(j+1-d(n+1))} \, d\sigma \right)^{1/2}$$

If we choose n large enough such that $j + 1 - d(n+1) < -1/2$ it follows that the right member of preceding expression is bounded by $C_j \|\varphi\|_{L^2}$. As regards of second term on the right in (11), we can show easily that

$$(15) \qquad \left| u\left(\rho(t) \int_{|\sigma| \le C_m} e^{it\sigma} \widehat{D^j \varphi}(\sigma) \, d\sigma \right) \right| \le C_j' \|\varphi\|_{L^2}$$

for any $\varphi \in D(\Delta, D_{A^*})$. To deduce a similar estimate for

$$\left| u\left(D_t^1 \rho(t) \int_{|\sigma| \ge C_m} e^{it\sigma} E(t,\sigma) \widehat{D^j \varphi}(\sigma) \, d\sigma \right) \right|$$

we put

$$f_+^j(t) = \begin{cases} \displaystyle\int_{|\sigma| \ge C_m} e^{it\sigma} D_t^1 \rho(t) E(t,\sigma) \widehat{D^j \varphi}(\sigma) \, d\sigma & t \ge 0 \\ 0 & t < 0 \end{cases}$$

and

$$f_-^j(t) = \begin{cases} 0 & t \ge 0 \\ \displaystyle\int_{|\sigma| \ge C_m} e^{it\sigma} D_t^1 \rho(t) E(t,\sigma) \widehat{D^j \varphi}(\sigma) \, d\sigma & t < 0. \end{cases}$$

Obviously $f_+^j, f_-^j \in D(\Delta, H)$. Hence,

$$\left| u\left(D^1 \rho(t) \int_{|\sigma| \geq C_m} e^{it\sigma} E(t,\sigma) \widehat{D^j \varphi}(\sigma)\, d\sigma\right)\right| \leq \tag{16}$$

$$\leq C \sum_{p=0}^{l} \sup \left(\| D^p f_+^j(t)\| + \| D^p f_-^j(t)\|\right).$$

Let m be an arbitrary positive integer. For every real σ with $|\sigma| \geq C_m$, denote by $q_m(\sigma)$ the smallest non-negative number such that $q_m(\sigma) = m \log |\sigma + iq_m(\sigma)|$. After suitable deformation of contours in the complex plane, the functions $f_+^j(t)$ and $f_-^j(t)$ can be expressed in the following form

$$f_+^j(t) = \int_{\Gamma_m^+} e^{it\lambda} D_t^1 \rho(t) \lambda^j E(t,\lambda) \hat{\varphi}(\lambda)\, d\lambda \tag{17}$$

$$f_-^j(t) = \int_{\Gamma_m^-} e^{it\lambda} D_t^1 \rho(t) \lambda^j E(t,\lambda) \hat{\varphi}(\lambda)\, d\lambda \tag{18}$$

where $\Gamma_m^+ = \{\lambda ; \lambda = \sigma + iq_m(\sigma) ; |\sigma| \geq C_m\}$ and $\Gamma_m^- = \{\lambda ; \lambda = \sigma - iq_m(\sigma) ; |\sigma| \leq C_m\}$. All the integrands in the above are holomorphic functions of λ in Σ_m. From (5) it is easily to see that the shift of the integration is legitimate. In virtue of (8),

$$\| D_t^j E(t,\lambda)\| \leq C_j^1 |\lambda|^{M(n+1)} \exp(N(n+1)|Im\lambda|), \qquad \lambda \in \Sigma_m,\ t \in \Delta.$$

Thus from Paley-Wiener Theorem we get

$$\| D^p f_+^j(t)\| \leq C_j^1 \int \exp(q_m(\sigma)(N(n+1) - \varepsilon'')) |\lambda|^{j+p+Mn} |d\lambda| \cdot \|\varphi\|_{L^2} \tag{19}$$

where $\varepsilon'' = \varepsilon - \varepsilon'$ and $\varepsilon > (k + l + 1/2) d^{-1} N + N$. We may find a non-negative integer n such that

$$d^{-1}(k + l + 1/2) + 1 < n + 1 < N^{-1} \varepsilon.$$

Choosing $\varepsilon' = 2^{-1}(\varepsilon - N(n+1))$ and m so large such that $Mn + p + k - 2^{-1} m \varepsilon' < 2^{-1}$ we deduce from (19)

$$\| D^p f_+^j(t)\| \leq C_j^1 \|\varphi\|_{L^2} \qquad \text{for} \quad 0 \leq j \leq k,\ 0 \leq p \leq 1 \tag{20}$$

We can show in a similar way

$$\| D^p f_-^j(t)\| \leq C_j^2 \|\varphi\|_{L^2} \qquad 0 \leq j \leq k,\ 0 \leq p \leq 1. \tag{21}$$

Collecting these results and using (16) we get

$$\left| u\left(D_t^1 \rho(t) \int_{|\sigma| \geq C_m} e^{it\sigma} E(t,\sigma) \widehat{D^j \varphi}(\sigma)\, d\sigma\right)\right| \leq C_j \|\varphi\|_{L^2}, \qquad 0 \leq j \leq k. \tag{22}$$

for any $\varphi \in D(\Delta, D_{A^*})$. Now combining (13), (15) and (22) we obtain

$$(23) \qquad |D^j u(\varphi)| \le M_j \|\varphi\|_{L^2} \quad \text{for} \quad 0 \le j \le k \quad \text{and} \quad \varphi \in D(\Delta, D_{A^*}).$$

Since the space $D(\Delta, D_{A^*})$ is dense in $D(\Delta, H)$, the inequality (23) is obviously true for every $\varphi \in D(\Delta, H)$. From Hanch-Banach theorem it follows that $D^j u \in L^2(\Delta, H)$ for $0 \le j \le k$. Taking inverse Fourier transform, this implies that $u \in C^{k-1}(\Delta, H)$.

Remark. Suppose that $d > 1$. From the proof it is easily to see that the preceding is true if we merely assume that $f \in C^{k-[d-1]}$.

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