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## On Decompositions of Matrix Spaces with Applications to Matrix Equations

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**Algebra lineare.** — *On Decompositions of Matrix Spaces with Applications to Matrix Equations* (*). Nota (**) di ADI BEN-ISRAEL, presentata dal Socio B. SEGRE.

RIASSUNTO. — Si dimostra un teorema di decomposizione dello spazio $C^{m\times n}$, delle matrici complesse $m\times n$, in una somma diretta di sottospazi ortogonali complementari, come conseguenza di un teorema corrispondente dello spazio vettoriale di dimensione $mn$. Si danno inoltre delle applicazioni per equazioni matriciali.

## INTRODUCTION.

A theorem on the decomposition of $C^{m\times n}$, the space of $m\times n$ complex matrices, into a direct sum of orthogonal complementary subspaces [1] is proved here as a consequence of the corresponding theorem in $C^{mn}$, the $mn$ dimensional complex vector space. Applications to matrix equations are given.

## § 0. – NOTATIONS.

$C^n$ the $n$-dimensional complex vector space

$(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i$ the standard inner product in $C^n$

$\|x\|_2 = (x, x)^{1/2}$ the Euclidean norm in $C^n$.

For any subspace L of $C^n$:

$L^\perp$ the orthogonal complement of L

$C^n = L \oplus M$ denotes $M = L^\perp$

$C^{m\times n}$ the space of $m\times n$ complex matrices.

For any $A \in C^{m\times n}$:

$A^t$ the transpose of A

$A^*$ the conjugate transpose of A

$A^+$ the generalized inverse of A. [6]

R (A) the range of A

N (A) the null space of A.

For any subspace L of $C^n$:

$P_L$ the perpendicular projection on L

i.e. $P_L = P_L^2 = P_L^*$ , $R(P_L) = L$.

For any $A \in C^{m\times n}$ , $B \in C^{p\times q}$ the Kronecker product of A, B is

$A \otimes B = (a_{ij} B) \in C^{mp\times nq}$ $(i = 1, \cdots, m\,;\ j = 1, \cdots, n)$, [5].

If not specified, the dimensions of matrices should be clear from the context.

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## § 1. – A correspondence between $C^{m \times n}$ and $C^{mn}$.

Aside from the practical aspect of representing matrices on tapes or punched cards, there seems to be little interest or use in the observation that any $m \times n$ matrix may be regarded as an $mn$-dimensional vector. While most of the interesting matrix properties are lost in passing from $C^{m \times n}$ to $C^{mn}$, those vector properties of linearity, convexity, standard inner product and the Euclidean norm are naturally preserved by the following correspondence.

*Definition 1*: Let $v : C^{m \times n} \to C^{mn}$ be the mapping assigning to any

$$X = (x_{ij}) \in C^{m \times n} \quad \text{the vector} \quad v(X) = (v_k), \quad (k = 1, \cdots, mn),$$

given by

$$v_{n(i-1)+j} = x_{ij} \qquad (i = 1, \cdots, m; j = 1, \cdots, n)$$

i.e. $v(X)$ is the vector obtained by reading the rows of X one by one.

The mapping $v$ induces in $C^{m \times n}$ the inner product

$$(X, Y) = (v(X), v(Y)) = \sum_{i,j} x_{ij} \bar{y}_{ij} = \text{trace } Y^* X \tag{1}$$

and the norm

$$\| X \| = \| v(X) \|_2 = \Big( \sum_{i,j} \| x_{ij} \|^2 \Big)^{1/2}. \tag{2}$$

Since $v : C^{m \times n} \to C^{mn}$ is a nonsingular linear transformation it is clear that L is a subspace of $C^{m \times n}$ if and only if $v(L)$ is a subspace of $C^{mn}$, and that $\dim L = \dim v(L)$. The following subspaces in $C^{m \times n}$ are of special interest:

*Definition 2*: For any $A \in C^{m \times p}$, $B \in C^{q \times n}$

$$R(A, B) = \{X : X = AYB \quad \text{for some } Y \in C^{p \times q}\} \tag{3}$$

the *range* of (A, B)

$$N(A, B) = \{Y : AYB = 0\} \tag{4}$$

the *null space* of (A, B).

The vector space counterparts of these subspaces are given in:

LEMMA 1:

(i) $v(R(A, B)) = R(A \otimes B^t)$

(ii) $v(N(A, B)) = N(A \otimes B^t)$.

*Proof*.—Follows from the easily verified

$$v(AYB) = (A \otimes B^t)\, v(Y), \quad \text{for all } Y, \quad \text{e.g. [5] p. 9.} \tag{5}$$

Not all the subspaces in $C^{m \times n}$ are of the form (3) or (4) since not all the $mn \times pq$ matrices are of the form $A \otimes B'$, $A \in C^{m \times p}$, $B \in C^{q \times n}$. For example, the subspace of symmetric matrices in $R^{n \times n}$ can be represented in the form (3) but only after rearrangement of components.

## § 2. – On range-null space decompositions of $C^{m \times n}$.

For any $A \in C^{m \times n}$ we recall that

(6) $$C^m = R(A) \oplus N(A^*)$$

The analogous result in $C^{m \times n}$ is:

THEOREM 1 ([1], [3]).—*For any* $A \in C^{m \times p}$, $B \in C^{q \times m}$

(7) $$C^{m \times n} = R(A, B) \oplus N(A^*, B^*).$$

*Proof.*—Follows from lemma 1 since, by (6), the subspaces

$$R(A \otimes B^t) = v(R(A, B))$$

and

$$N((A \otimes B^t)^*) = N(A^* \otimes B^{*t}) = v(N(A^*, B^*))$$

are orthogonal complements in $C^{mn}$.

This theorem may be stated more generally [1], but the restriction to matrices makes possible the above elementary derivation.

Before giving the perpendicular projections corresponding to the decomposition (7) we need:

LEMMA 2.—*Let* $S, T, S_i, T_i$ $(i = 1, \cdots, k)$ *be matrix spaces and let*

$$f : \prod_{i=1}^{k} S_i \to S, : \prod_{i=1}^{k} T_i \to T$$

*be a mapping satisfying:*

(i) *For all* $A_i \in S_i$, $B_i \in T_i$ $(i = 1, \cdots, k)$

$$f(A_1, \cdots, A_k) f(B_1, \cdots, B_k) = f(C_1, \cdots, C_k)$$

*where for* $i = 1, \cdots, k$

$$C_i = A_i B_i \quad \text{or} \quad B_i A_i \text{ (1)}$$

(ii) *If* $A_i$ $(i = 1, \cdots, k)$ *are Hermitian then so is* $f(A_1, \cdots, A_k)$.

*Then*:

(8) $$(f(A_1, \cdots, A_k))^+ = f(A_1^+, \cdots, A_k^+)$$

*for all*

$$A_i \in S_i \qquad (i = 1, \cdots, k).$$

*Proof.*—The right side of (8) satisfies the defining conditions of the generalized inverse of $f(A_1, \cdots, A_k)$, e.g. [6].

(1) One choice for each i but possibly different choices for different $i$, e.g. $f(A_1, A_2) f(B_1, B_2) = f(A_1 B_1, B_2 A_2)$.

COROLLARY 1.—*For any matrices* A , B.

(i) $(A^t)^+ = (A^+)^t$ (2)
(ii) $(A \otimes B)^+ = A^+ \otimes B^+$

*Proof.*—Use lemma 2 with:

(i) $k = 1 , f(A) = A^t$
(ii) $k = 2 , f(A , B) = A \otimes B$

and verify in each case that $f$ satisfies conditions (i), (ii) of lemma 2.

COROLLARY 2.—*The perpendicular projections of* $C^{m \times n}$ *on the subspaces* R (A , B) , N (A*, B*) *of theorem* 1 *are given by*:

$$P_{R(A,B)} X = AA^+ XB^+ B \tag{9}$$

$$P_{N(A^*,B^*)} X = X - AA^+ XB^+ B \tag{10}$$

*for any* $X \in C^{m \times n}$

*Proof.*—(9) follows from (5) and lemma 1 since

$$\begin{aligned} P_{R(A \otimes B^t)} &= (A \otimes B^t)(A \otimes B^t)^+ , && \text{e.g. [3]} \\ &= (A \otimes B^t)(A^+ \otimes B^{+t}) , && \text{by corollary 1} \\ &= (AA^+) \otimes (B^+ B)^t , && \text{e.g. [5]} \end{aligned}$$

(10) follows now from (9) and (7).

The projection (9) is rewritten as

$$P_{R(A,B)} X = P_{R(A)} X P_{R(B^*)}$$

and (10), by subtracting and adding $AA^+ X$, becomes

$$P_{N(A^*,B^*)} X = P_{N(A^*)} X + P_{R(A)} X P_{N(B)}$$

or alternatively

$$P_{N(A^*,B^*)} X = P_{N(A^*)} X P_{R(B^*)} + X P_{N(B)} .$$

The corresponding projections in $C^{mn}$ are therefore

$$P_{v(R(A,B))} = P_{R(A \otimes B^t)} = P_{R(A)} \otimes P^t_{R(B^*)}$$

$$\begin{aligned} P_{v(N(A^*,B^*))} &= P_{N(A^* \otimes B^{*t})} = P_{N(A^*)} \otimes I + P_{R(A)} \otimes P^t_{N(B)} \\ &= P_{N(A^*)} \otimes P^t_{R(B^*)} + I \otimes P^t_{N(B)} . \end{aligned}$$

(2) This is different from

$$(A^*)^+ = (A^+)^*, [6]$$

which also can be proved by lemma 2.

## § 3. – Applications.

The above results have direct applications to matrix equations:

Theorem 2 (Penrose [6]).—*The matrix equation*

$$AXB = C \tag{11}$$

*is solvable if, and only if*

$$AA^+CB^+B = C \tag{12}$$

*in which case the general solution is*

$$A^+CB^+ + Y - A^+AYBB^+ \,,\ Y \text{ arbitrary} \tag{13}$$

*Proof.*—(11) is solvable if, and only if $C \in R(A, B)$ which proves (12) by using (9). The general solution is any particular solution, e.g. $A^+CB^+$ by (12), plus the general element of $N(A, B)$ which by (10) proves (13). The least squares solution of (11) are also easily obtainable from the above results:

Theorem 3 (Penrose [7]).—*The matrix*

$$A^+CB^+ \tag{14}$$

*is of minimal norm* (2) *among all matrices minimizing*

$$\|AXB - C\|.$$

*Proof.*—It follows from the corresponding result in $C^{mn}$ that the vector $(A \otimes B')^+ v(C) = (A^+ \otimes B^{+\prime}) v(C) = v(A^+CB^+)$ is of minimal norm among all vectors minimizing

$$\|(A \otimes B') v(X) - v(C)\| = \| v(AXB - C)\| .$$

The following characterization of $A^+$ is also interesting:

Corollary 3.—*Let* $A \in C^{m \times n}$ *and* X *satisfy*

$$AXA = A. \tag{15}$$

*Then the following are equivalent*:

(i) $X = A^+$

(ii) $X \in R(A^*, A^*)$

(iii) X is the minimal norm (2) solution of (15).

*Proof.*—The general solution of (15) is

$$\begin{aligned} X &= A^+AA^+ + Y - A^+AYAA^+ \quad , && \text{by (13)} \\ &= A^+ + P_{N(A,A)}\, Y \quad , && \text{by (10)} \end{aligned} \tag{16}$$

(i) $\Longleftrightarrow$ (ii) now follows from (7) since

$$A^+ = A^+AA^+AA^+ = A^*A^{*+}A^+A^{*+}A^* \in R(A^*, A^*)$$

and (i) $\Longleftrightarrow$ (iii) from

$$\|X\|^2 = \|A^+\|^2 + \|P_{N(A,A)}Y\|^2 \qquad \text{in (16).}$$

An application to matrix inequalities will now be given. For any $X = (x_{ij}) \in R^{m\times n}$ we denote by $X \geqq 0$ the fact

$$x_{ij} \geq 0 \; (i = 1, \cdots, m \; ; \; j = 1, \cdots, n).$$

COROLLARY 4.—*Let* A , B , C *be real matrices. Then the system of equations and inequalities*

$$\text{(17)} \qquad AXB = C \quad , \quad X \geqq 0$$

*is solvable if, and only if,*

$$\text{(18)} \qquad A^t U B^t \geqq 0 \qquad \text{implies: trace } U^t C \geqq 0.$$

*Proof.*—The solvability of (17) is equivalent to that of

$$(A \otimes B^t)\, v(X) = v(C) \quad , \quad v(X) \geqq 0$$

which by Farkas' theorem [4] is equivalent to:

$$(A \otimes B^t)^t\, v(U) \geqq 0 \qquad \text{implies} \quad (v(U), v(C)) \geq 0$$

or, by (5) and (1), to (18).

Applications of these results to iterative methods of generalized inversion are given in [10]. In particular it is shown that for $X_0 \in R(A^*, A^*)$ the iterative method [2] (or the higher order methods of [8], [9]):

$$X_{k+1} = X_k(2I - AX_k) \qquad (k = 0, 1, \cdots)$$

converges to $A^+$ if, and only if the spectral radius:

$$\rho\,(P_{R(A)} - AX_0) < 1,$$

but that it may diverge for $X_0$ with $P_{N(A,A)} X_0 \neq 0$ even if

$$\rho\,(P_{R(A)} - AX_0) = 0$$

$$\text{e.g. } A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \; X_0 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \; \epsilon \neq 0.$$

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