Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Ivan Erdelyi

Partial Isometries Defined by a Spectral Property on Unitary Spaces

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **44** (1968), n.6, p. 741–747.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1968_8_44_6_741_0>

L'utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l'utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

Articolo digitalizzato nel quadro del programma bdim (Biblioteca Digitale Italiana di Matematica) SIMAI & UMI

http://www.bdim.eu/

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1968.

Algebra lineare. — *Partial Isometries Defined by a Spectral Property on Unitary Spaces.* Nota di IVAN ERDELYI, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si dimostra che, in un qualunque spazio di dimensione finita, lo spettro di una isometria parziale A è un sottoinsieme dell'unione della circonferenza unitaria con l'origine se, e soltanto se, l'aggiunta A* di A risulta permutabile con una potenza intera e positiva di A. Si dà inoltre una proprietà moltiplicativa delle isometrie parziali definite da tale condizione spettrale.

INTRODUCTION.

Whenever a partial isometry is not invertible, its spectrum is a subset of the closed unit disk (for further properties of the spectrum see e.g. [1]). The inclusion of the singleton (0) in the spectrum is imposed by the singularity of the operator; nevertheless, the presence of spectral numbers elsewhere inside the unit circle is not always necessary.

What conditions must a partial isometry satisfy in order that its spectrum be a subset of the union of the unit circle and (0)?

It is the purpose of this paper to investigate for the most general class of partial isometries, on a finite-dimensional unitary space E^n, whose eigenvalues λ enjoy the spectral property

$$|\lambda| = 0, \quad \text{or } 1. \tag{1}$$

For an operator A, we denote by A^*, $R(A)$, $N(A)$, and $\rho(A)$ the adjoint, the range, the null space, and the rank, respectively, of A. The norm $\|A\|$ of A is defined by $\|A\| = \operatorname{Sup}\{\|Ax\| : \|x\| = 1\}$, conformable with the vector norm $\|x\|$ of the E^n space, under consideration. We write $S^{\perp}$ for the orthogonal complement of a subspace S, and $\mathcal{P}_S$ for the orthogonal projection on S.

A partial isometry A acts isometrically on the orthogonal complement $N(A)^{\perp} = R(A^*)$ of its null space $N(A)$,

$$\|Ax\| = \|x\|, \qquad x \in R(A^*),$$

yet the union of these subspaces does not cover the entire space unless A is either zero or invertible.

An eigenvector x of a partial isometry A may have its length conserved, annihilated, or contracted by the eigenequation

$$Ax = \lambda x,$$

(*) Nella seduta dell'8 giugno 1968.

according to the location of x in E^n, as follows [2]:

(i) $|\lambda| = 1$, iff (= if and only if) $x \in R(A^*)$;

(ii) $\lambda = 0$, iff $x \in N(A)$;

(iii) $0 < |\lambda| < 1$, iff $x \notin R(A^*) \cup N(A)$.

By the spectral condition (1), foregoing case (iii) is avoided.

Some further properties of partial isometries are referred to in the following discussion.

Each of the following conditions is necessary and sufficient that A be a partial isometry:

$$A = AA^*A \quad , \quad A^* = A^*AA^* \quad , \quad A^*A = \mathcal{P}_{R(A^*)} \quad , \quad AA^* = \mathcal{P}_{R(A)} . \tag{2}$$

From metrical standpoint, a partial isometry A and its natural powers A^m are contractions,

$$\|A\| \leq 1 \quad , \quad \|A^m\| \leq \|A\|^m \leq 1 .$$

Any partial isometry A, of rank $r > 0$, can be represented in the factorized matrix form [3]:

$$A = U \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} V, \tag{3}$$

where U and V are unitary, and I_r denotes the r–dimensional identity.

Quasi-commuting partial isometries.

We call a partial isometry A quasi-commuting if, together with A^*, it satisfies the following quasi-commuting relation [4]:

$$(AA^* - A^*A)\,A^m = A^m (AA^* - A^*A), \quad \text{for some natural number } m. \tag{4}$$

With the help of relations (2), it is easy to show that, for a partial isometry A, the foregoing relation (4) is equivalent to

$$A^*A^m = A^m A^*. \tag{5}$$

Relation (4) appears as a weakened normality condition.

As a first consequence of the spectral property (1), we may consider the following

LEMMA 1. *If* A *is a contraction that satisfies condition* (1), *then to the k nonzero eigenvalues of* A, *each counted with its algebraic multiplicity, there correspond k linearly independent eigenvectors.*

Proof.—If the assertion of the Lemma is not true then there exists a $p\,(> 1)$—dimensional subspace S of E^n, such that S is invariant under A, and $A - \lambda I = N$ is nilpotent of index p on S. Here, λ denotes a nonzero eigenvalue of A, and I is the identity operator.

Furthermore, for any $x \in S$, such that $N^{p-1} x \neq 0$, the set

$$x, Nx, N^2 x, \cdots, N^{p-1} x$$

forms a basis for S. Then, for $m \geq p - 1$,

$$A^m x = (\lambda I + N)^m x = \lambda^m x + \binom{m}{1} \lambda^{m-1} Nx + \cdots + \binom{m}{p-1} \lambda^{m-p+1} N^{p-1} x.$$

Since $|\lambda| = 1$, as m increases the norm of $A^m x$ exceeds any bound. But A is a contraction, and then

$$\| A^m x \| \leq \| A^m \| \cdot \| x \| \leq \| x \|.$$

Thus it follows that the statement of the Lemma is true.

If the spectral condition (1) is strengthened by a rank condition, then a contraction becomes a normal partial isometry, as it follows from

LEMMA 2.—*A contraction* A *is a normal partial isometry on* E^n *iff*

(i) $|\lambda(A)| = 0$, or 1, *and*

(ii) $\rho(A) = \rho(A^2)$ (*or equivalently*, $R(A) \cap N(A) = (0)$).

Proof.—If [1]: Since A is a contraction, $I - A^* A$ is positive semidefinite.

In view of condition (i), for all eigenvectors x, associated with nonzero eigenvalues of A, we have

$$(x, x) = (Ax, Ax) = (A^* Ax, x),$$

and then

$$((I - A^* A) x, x) = 0.$$

Since $I - A^* A \geq 0$, there follows

$$(I - A^* A) x = 0. \tag{6}$$

By condition (ii) of the Lemma, the rank $r = \rho(A)$ equals the dimensions of both $R(A)$ and $R(A^*)$. Since the nilpotent part of A is the zero operator, Lemma 1 requires that the eigenvectors, corresponding to the nonzero eigenvalues of A, form a basis for both $R(A)$ and $R(A^*)$. Consequently, $R(A) = R(A^*)$, and then A is an EPr operator [5, 6].

On the other hand, the r linearly independent eigenvectors x of A, associated with the eigenvalues of modulus 1, satisfy (6). Moreover, any basis of $N(A)$ provides $(n - r)$ linearly independent eigenvectors for A, annulled by $A^* A$. Thus, it follows that $A^* A = \mathcal{P}_{R(A^*)}$, and hence A is a partial isometry. An EPr partial isometry is normal [3, 6].

Only if: This a straightforward consequence of the fact that every normal partial isometry is unitarily similar to the direct sum of a unitary and a zero operator.

(1) This version of the proof was suggested to the author by J. Z. Hearon.

We now prove our main

THEOREM 1.—*The spectrum of a partial isometry* A *on* E^n *is a subset of the union of the unit circle and the singleton* (0) *iff* A^* *commutes with* A^m, *for some natural* m.

Proof.—If: We follow C. M. Price [7], in the proof of a spectral property for the generalized inverse of matrices.

For any nonzero eigenvalue λ of A,

$$Ax = \lambda x \quad , \quad x = \lambda^{-1} Ax = \lambda^{-m} A^m x,$$

and then, by using (5) and the first of (2), we have successively:

$$A^* x = \lambda^{-m} A^* A^m x = \lambda^{-m} A^m A^* x = \lambda^{-m-1} A^m A^* Ax = \lambda^{-m-1} A^m x = \lambda^{-1} x.$$

Since both A and A^* are partial isometries, we must simultaneously have $|\lambda| \leq 1$, $|\lambda|^{-1} \leq 1$, and hence $|\lambda| = 1$.

Only if: For any linear operator A, the space E_n splits into the direct sum

$$E^n = R(A^m) \oplus N(A^m), \qquad \text{for some natural } m,$$

A is nonsingular on $R(A^m)$ and nilpotent of index $\leq m$ on $N(A^m)$, and both $R(A^m)$ and $N(A^m)$ are invariant under A. If A is a partial isometry, A^m is a contraction, and if moreover A is supposed to have all its nonzero eigenvalues of modulus 1, then all its powers have the same property.

Thus, by the hypotheses of the Theorem, the contraction A^m enjoys the following properties:

(i) $|\lambda(A^m)| = 0$, or 1;

(ii) $\rho(A^m) = \rho(A^{2m})$.

The contraction A^m, satisfying the premises of Lemma 2, is normal. Then, as a trivial case of Fuglede's theorem, it follows condition (5).

COROLLARY.—*A quasi-commuting partial isometry is the direct sum of a unitary and a nilpotent operator of index* $\leq m$.

In particular, any of these summands might be absent.

Thus, the most general partial isometries which satisfy the spectral condition (1) are the quasi-commuting partial isometries. These, as shown in [4], are related to the quasi-commuting inverses A^- of A, by the functional equation $A^* = A^-$.

A MULTIPLICATIVE PROPERTY.

An equivalent defining property for a quasi-commuting partial isometry A may be given in terms of the decomposition (3), by the following

LEMMA 3.—*Let*

$$A = U \begin{bmatrix} I_r & \\ & O \end{bmatrix} V \quad \textit{and} \quad VU = \begin{bmatrix} Q & \cdots \\ \cdots & \cdots \end{bmatrix}$$

be the factorized form of a partial isometry A, *of rank* $r > 0$, *and the conformable partitioned form of the product* VU, *respectively.* U *and* V *are unitary, and the* r *by* r *submatrix* Q *is not necessarily nonsingular.*

The partial isometry A *is quasi-commuting iff*

$$\text{(7)} \qquad \mathrm{VU}\begin{bmatrix} \mathrm{Q}^{m-1} & \\ & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{Q}^{m-1} & \\ & 0 \end{bmatrix}\mathrm{VU}.$$

Proof.—By performing the *mth* power of A, we obtain

$$\mathrm{A}^m = \mathrm{U}\begin{bmatrix} \mathrm{I}_r & \\ & 0 \end{bmatrix}\mathrm{VU}\begin{bmatrix} \mathrm{I}_r & \\ & 0 \end{bmatrix}\mathrm{V}\cdots\mathrm{U}\begin{bmatrix} \mathrm{I}_r & \\ & 0 \end{bmatrix}\mathrm{V} = \mathrm{U}\begin{bmatrix} \mathrm{Q}^{m-1} & \\ & 0 \end{bmatrix}\mathrm{V}.$$

We then have

$$\text{(8)} \qquad \mathrm{A}^m\,\mathrm{A}^* = \mathrm{U}\begin{bmatrix} \mathrm{Q}^{m-1} & \\ & 0 \end{bmatrix}\mathrm{U}^* \quad , \quad \mathrm{A}^*\,\mathrm{A}^m = \mathrm{V}^*\begin{bmatrix} \mathrm{Q}^{m-1} & \\ & 0 \end{bmatrix}\mathrm{V}.$$

If A is supposed to be quasi-commuting, then by equating the right-hand sides of (8), we obtain condition (7).

Conversely, condition (7), with the help of (8) turns into

$$\mathrm{A}^*\,\mathrm{A}^m = \mathrm{A}^m\,\mathrm{A}^*.$$

We note that if the submatrix Q of the partitioned form of VU is the zero matrix, then $\mathrm{A}^2 = 0$, and the conclusion of Lemma 3 is trivial.

Now, we answer the question: When is the product of two partial isometries of rank r a quasi-commuting partial isometry of the same rank r?

THEOREM 2.—*Let* A *and* B *be two partial isometries of rank* r, *given in form* (3),

$$\text{(9)} \qquad \mathrm{A} = \mathrm{U}\begin{bmatrix} \mathrm{I}_r & \\ & 0 \end{bmatrix}\mathrm{V} \quad , \quad \mathrm{B} = \mathrm{U}_1\begin{bmatrix} \mathrm{I}_r & \\ & 0 \end{bmatrix}\mathrm{V}_1, \textit{ with } \mathrm{U}, \mathrm{V}, \mathrm{U}_1, \textit{ and } \mathrm{V}_1 \textit{ unitary},$$

and let denote the conformable partitioned forms of the products VU_1, *and* $\mathrm{V}_1\mathrm{U}$ *by*

$$\mathrm{VU}_1 = \begin{bmatrix} \mathrm{Q}_1 & \cdots \\ \cdots & \cdots \end{bmatrix} \quad , \quad \mathrm{V}_1\mathrm{U} = \begin{bmatrix} \mathrm{Q} & \cdots \\ \cdots & \cdots \end{bmatrix}.$$

The product

$$\mathrm{P} = \mathrm{AB}$$

is a quasi-commuting partial isometry of rank r *iff the following conditions are satisfied*:

(i) VU_1 *is block-diagonal*, $\mathrm{VU}_1 = \begin{bmatrix} \mathrm{Q} & \\ & \cdots \end{bmatrix}$;

(ii) $\mathrm{V}_1\mathrm{U}\begin{bmatrix} (\mathrm{QQ}_1)^{m-1} & \\ & 0 \end{bmatrix} = \begin{bmatrix} (\mathrm{Q}_1\mathrm{Q})^{m-1} & \\ & 0 \end{bmatrix}\mathrm{V}_1\mathrm{U}$, *for some natural* m.

Proof.—Property (i) is a necessary and sufficient condition that the product P be a partial isometry of rank r. In fact, from (9) we obtain

$$P = U \begin{bmatrix} Q & \\ & O \end{bmatrix} V_1 \quad , \quad PP^* P = U \begin{bmatrix} QQ^* Q & \\ & O \end{bmatrix} V_1 ,$$

and since P is supposed to have rank r, Q is nonsingular. Then from $Q = QQ^* Q$, it follows that Q is unitary, and then a direct computation shows that VU_1 is block-diagonal. Conversely, if VU_1 is block-diagonal, Q must be unitary and then P is a partial isometry. Such condition (i) may be deduced from some general closure conditions for partial isometries under multiplication, e.g. [3, 8]. The product P, with the help of the unitary

$$U_2 = U \begin{bmatrix} Q & \\ & I_{n-r} \end{bmatrix} \tag{10}$$

assumes the form (3):

$$P = U_2 \begin{bmatrix} I_r & \\ & O \end{bmatrix} V_1 , \tag{11}$$

where

$$V_1 U_2 = V_1 U \begin{bmatrix} Q & \\ & I_{n-r} \end{bmatrix} = \begin{bmatrix} Q_1 Q & \cdots \\ \cdots & \cdots \end{bmatrix} .$$

By Lemma 3, the partial isometry P, as given in (11), is quasi-commuting iff

$$V_1 U_2 \begin{bmatrix} (Q_1 Q)^{m-1} & \\ & O \end{bmatrix} = \begin{bmatrix} (Q_1 Q)^{m-1} & \\ & O \end{bmatrix} V_1 U_2 . \tag{12}$$

Condition (12), with the help of (10), becomes successively:

$$V_1 U \begin{bmatrix} Q (Q_1 Q)^{m-1} & \\ & O \end{bmatrix} = \begin{bmatrix} (Q_1 Q)^{m-1} & \\ & O \end{bmatrix} V_1 U \begin{bmatrix} Q & \\ & I_{n-r} \end{bmatrix} ,$$

$$V_1 U \begin{bmatrix} Q (Q_1 Q)^{m-1} Q^{-1} & \\ & O \end{bmatrix} = \begin{bmatrix} (Q_1 Q)^{m-1} & \\ & O \end{bmatrix} V_1 U ,$$

thus, condition (ii) of the Theorem follows.

Conversely, condition (ii) with the help of (10) may be transformed in (12). The proof is complete [2].

(2) *Acknowledgment.* This work has been carried out as part of the research program of the Kansas State University in Manhattan, Kansas.

The author wishes to express his appreciation to John Z. Hearon for many helpful suggestions on a previous version of the paper.

REFERENCES.

[1] P. R. HALMOS and J. E. MCLAUGHLIN, *Partial isometries*, « Pacific J. Math. », *13*, 585–596 (1963).

[2] I. ERDELYI, *On partial isometries in finite-dimensional Euclidean spaces*, « J. SIAM Appl. Math. », *14*, 453–467 (1966).

[3] J. Z. HEARON, *Partially isometric matrices*, « J. Res. Nat. Bur. Standards Sect. B », *71*, 225–228 (1967)

[4] I. ERDELYI, *The quasi-commuting inverses for a square matrix*, « Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. », (8), *42*, 626–633 (1967).

[5] I. J. KATZ and M. H. PEARL, *On EPr and normal EPr matrices*, « J. Res. Nat. Bur. Standards Sect. B », *70*, 47–77 (1966).

[6] T. S. BASKETT and I. J. KATZ, *Theorems on products of EPr matrices*, « J. Linear Algebra Appl. », in press.

[7] C. M. PRICE, *The matrix pseudoinverse and minimal variance estimates* « SIAM Rev. », *6*, 115–120 (1964).

[8] I. ERDELYI, *Partial isometries closed under multiplication on Hilbert spaces*, « J. Math. Anal. Appl. », in press.