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On the non-linear wave equation with dissipative term discontinuous with respect to the velocity. Nota II

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta dell'11 maggio 1968

Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi matematica. — *On the non-linear wave equation with dissipative term discontinuous with respect to the velocity.* Nota II di LUIGI AMERIO e GIOVANNI PROUSE, presentata (*) dal Corrisp. L. AMERIO.

SUNTO. — Si dimostra il teorema di esistenza enunciato nel § 1 della Nota I.

3. – We now consider the problem of the existence of a solution of (1.9), (1.14). The existence theorem will be proved at first under supplementary conditions on the function $\beta(\eta)$; afterwards we shall treat the general case.

Assume that $u_0, u_1 \in H_0^1$, $Au_0 \in L^2$ *and that* $\beta(\eta)$, $\beta'(\eta)$ *are continuous and bounded on* R^1, *with* $\beta'(\eta) \geq 0$; *moreover that* $f(0) \in L^2$, $f'(t) \in L^1(0 \vdash\!\dashv T; L^2)$.

There exists then a solution $y(t) \in \Gamma$ *of* (1.9), (1.14).

a) For the proof we shall use the Galerkin–Faedo method (analogously to Lions and Strauss). In our case it is particularly useful to take as a "basis" the sequence $\{g_j\}$ of eigenfunctions of the operator A (already introduced at § 1).

Setting

$$y_n(t) = \sum_1^n{}_j \alpha_{nj}(t) g_j, \qquad (j = 1, \cdots, n) \tag{3.1}$$

we consider the system of "approximating equations" deducted from (1.9):

$$(y_n''(t), g_j)_{L^2} - (Ay_n(t), g_j)_{L^2} - (f(t), g_j)_{L^2} = -(\beta(y_n'(t)), g_j)_{L^2}, \tag{3.2}$$

(*) Nella seduta del 9 marzo 1968.

with the initial conditions

$$y_n(0) = \sum_1^n {}_j (u_0, g_j)_{L^2} g_j \qquad ((u_0, g_j)_{L^2} = \alpha_{nj}(0)), \tag{3.3}$$

$$y'_n(0) = \sum_1^n {}_j (u_1, g_j)_{L^2} g_j \qquad ((u_1, g_j)_{L^2} = \alpha'_{nj}(0)). \tag{3.4}$$

Hence

$$Ay_n(0) = -\sum_1^n {}_j \lambda_j (u_0, g_j)_{L^2} g_j . \tag{3.5}$$

From (3.3), (3.4), (3.5) it follows

$$\begin{aligned} &\| y_n(0) \|_{H_0^1} \leq \| u_0 \|_{H_0^1} \quad , \quad \| y'_n(0) \|_{L^2} \leq \| u_1 \|_{L^2} , \\ &\| y'_n(0) \|_{H_0^1} \leq \| u_1 \|_{H_0^1} \quad , \quad \| Ay_n(0) \|_{L^2} \leq \| Au_0 \|_{L^2} . \end{aligned} \tag{3.6}$$

Multiplying (3.2) by $\alpha'_{nj}(t)$ and adding, we obtain

$$(y''_n(t), y'_n(t))_{L^2} - (Ay_n(t), y'_n(t))_{L^2} - (f(t), y'_n(t))_{L^2} = -(\beta(y'_n(t)), y'_n(t))_{L^2} \tag{3.7}$$

that is (being $-(Ay_n(t), y'_n(t))_{L^2} = (y_n(t), y'_n(t))_{H_0^1}$, $(\beta(y'_n(t)), y'_n(t))_{L^2} \geq 0$)

$$\frac{1}{2} \frac{d}{dt} \| y_n(t) \|_E^2 \leq (f(t), y'_n(t))_{L^2}. \tag{3.8}$$

Hence

$$\| y_n(t) \|_E^2 \leq \| y_n(0) \|_E^2 + 2 \int_0^t \| f(\eta) \|_{L^2} \| y_n(\eta) \|_{L^2} \, d\eta , \tag{3.9}$$

from which, by (3.6) and (2.8),

$$\| y_n(t) \|_E^2 \leq 4 \left\{ \| y_n(0) \|_E^2 + \left(\int_0^T \| f(\eta) \|_{L^2} \, d\eta \right)^2 \right\} \leq \tag{3.10}$$

$$\leq 4 \left\{ \| u_0 \|_{H_0^1}^2 + \| u_1 \|_{L^2}^2 + \left(\int_0^T \| f(\eta) \|_{L^2} \, d\eta \right)^2 \right\} = K_1^2 ,$$

where $K_1 \geq 0$ does not depend on $\beta(\eta)$ and n.

b) Let us multiply (3.2) by $-\lambda_j \alpha_{nj}(t)$ and add the corresponding equations. Observing that $Ay'_n(t) = -\sum_1^n {}_j \lambda_j \alpha'_{nj}(t) g_j$, we obtain

$$\begin{aligned} (y''_n(t), Ay'_n(t))_{L^2} - (Ay_n(t), Ay'_n(t))_{L^2} - (f(t), Ay'_n(t))_{L^2} = \\ = -(\beta(y'_n(t)), Ay'_n(t))_{L^2} , \end{aligned} \tag{3.11}$$

that is

$$(3.12)\quad \frac{\mathrm{d}}{\mathrm{d}t}\{\|y'_n(t)\|^2_{H^1_0}+\|Ay_n(t)\|^2_{L^2}\}=-2(f(t),Ay'_n(t))_{L^2}+2(\beta(y'_n(t)),Ay'_n(t))_{L^2}.$$

Observe now that (by the hypotheses on $\beta(\eta)$) $g\in H^1_0\Rightarrow\beta(g)\in H^1_0$; hence

$$(3.13)\quad (\beta(y'_n(t)),Ay'_n(t))_{L^2}=-\int_\Omega\Big\{\beta'\Big(\frac{\partial y_n(t,x)}{\partial t}\Big)\sum_{j,k}^{1\ldots m}a_{jk}(x)\frac{\partial^2 y_n(t,x)}{\partial t\,\partial x_j}\frac{\partial^2 y_n(t,x)}{\partial t\,\partial x_k}+$$

$$+\beta\Big(\frac{\partial y_n(t,x)}{\partial t}\Big)a_0(x)\frac{\partial y_n(t,x)}{\partial t}\Big\}\,\mathrm{d}x\le 0.$$

Setting

$$(3.14)\quad \psi_n(t)=\{\|y'_n(t)\|^2_{H^1_0}+\|Ay_n(t)\|^2_{L^2}\}^{1/2},$$

it follows

$$(3.15)\quad \psi_n^2(t)\le\psi_n^2(0)-2\int_0^t(f(\eta),Ay'_n(\eta))_{L^2}\,\mathrm{d}\eta=$$

$$=\psi_n^2(0)-2(f(t),Ay_n(t))_{L^2}+2(f(0),Ay_n(0))_{L^2}+2\int_0^t(f'(\eta),Ay_n(\eta))_{L^2}\,\mathrm{d}\eta\le$$

$$\le\|u_1\|^2_{H^1_0}+\|Au_0\|^2_{L^2}+2\|f(0)\|_{L^2}\|Au_0\|_{L^2}+2\Big(\|f(0)\|_{L^2}+\int_0^T\|f'(\eta)\|_{L^2}\,\mathrm{d}\eta\Big)\psi_n(t)+$$

$$+2\int_0^t\|f'(\eta)\|_{L^2}\psi_n(\eta)\,\mathrm{d}\eta=M_1+2M_2\,\psi_n(t)+2\int_0^t\|f'(\eta)\|_{L^2}\psi_n(\eta)\,\mathrm{d}\eta,$$

where M_1, M_2 do not depend on $\beta(\eta)$ and n. Hence

$$(\psi_n(t)-M_2)^2\le M_1+M_2^2+2M_2\int_0^T\|f'(\eta)\|_{L^2}\,\mathrm{d}\eta+2\int_0^t\|f'(\eta)\|_{L^2}\,|\psi_n(\eta)-M_2|\,\mathrm{d}\eta,$$

that is, by (3.15),

$$(3.16)\quad \|y'_n(t)\|_{H^1_0}\le K_2\quad,\quad\|Ay_n(t)\|_{L^2}\le K_2,$$

where $K_2\ge 0$ is independent of $\beta(\eta)$ and n.

c) Let us now differentiate (3.2):

$$(3.17)\quad (y'''_n(t),g_j)_{L^2}-(Ay'_n(t),g_j)_{L^2}-(f'(t),g_j)_{L^2}=-(\beta'(y'_n(t))\,y''_n(t),g_j)_{L^2}.$$

Multiplying (3.17) by $\alpha''_{nj}(t)$ and adding, we obtain

$$(3.18)\quad (y'''_n(t),y''_n(t))_{L^2}+(y''_n(t),y'_n(t))_{H^1_0}=$$

$$=(f'(t),y''_n(t))_{L^2}-(\beta'(y'_n(t))\,y''_n(t),y''_n(t))_{L^2}$$

that is

$$(3.19)\quad \frac{d}{dt}\{\|y_n''(t)\|_{L^2}^2 + \|y_n'(t)\|_{H_0^1}^2\} = 2(f'(t), y_n''(t))_{L^2} - 2(\beta'(y_n'(t))\, y_n''(t), y_n''(t))_{L^2}.$$

Observing that $(\beta'(y_n')\, y_n'', y_n'')_{L^2} \geq 0$, it follows from (3.19)

$$(3.20)\quad \|y_n''(t)\|_{L^2}^2 + \|y_n'(t)\|_{H_0^1}^2 \leq 4\left\{\|y_n''(0)\|_{L^2}^2 + \|y_n'(0)\|_{H_0^1}^2 + \left(\int_0^T \|f'(\eta)\|_{L^2}\, d\eta\right)^2\right\} \leq$$

$$\leq 4\left\{\|y_n''(0)\|_{L^2}^2 + \|u_1\|_{H_0^1}^2 + \left(\int_0^T \|f'(\eta)\|_{L^2}\, d\eta\right)^2\right\}.$$

On the other hand

$$(3.21)\quad (y_n''(0), g_j)_{L^2} = (Ay_n(0), g_j)_{L^2} + (f(0), g_j)_{L^2} - (\beta(y_n'(0)), g_j)_{L^2},$$

that is

$$(3.22)\quad y_n''(0) = Ay_n(0) + \sum_1^n{}_k (f(0), g_k)_{L^2}\, g_k - \sum_1^n{}_k (\beta(y_n'(0)), g_k)_{L^2}\, g_k.$$

Bearing in mind (3.4) and the fact that $0 \leq \beta'(\eta) \leq K_\beta < +\infty$, it results

$$(3.23)\quad \int_\Omega \left(\beta\left(\frac{\partial y_n(0,x)}{\partial t}\right) - \beta(u_1(x))\right)^2 dx \leq K_\beta^2 \int_\Omega \left(\frac{\partial y_n(0,x)}{\partial t} - u_1(x)\right)^2 dx =$$

$$= K_\beta^2 \sum_{n+1}^{\infty}{}_k (u_1, g_k)_{L^2}^2.$$

Hence

$$(3.24)\quad \|\beta(y_n'(0))\|_{L^2} \leq \|\beta(u_1)\|_{L^2} + K_\beta \|u_1 - y_n'(0)\|_{L^2}.$$

By (3.20), (3.22), (3.24), it follows

$$(3.25)\quad \|y_n''(t)\|_{L^2}^2 + \|y_n'(t)\|_{H_0^1}^2 \leq 4\left\{\|u_1\|_{H_0^1}^2 + \left(\int_0^T \|f'(\eta)\|_{L^2}\, d\eta\right)^2 + 4\|Au_0\|_{L^2}^2 + \right.$$

$$\left. + 4\|f(0)\|_{L^2}^2 + 4\|\beta(u_1)\|_{L^2}^2 + 4K_\beta^2 \|u_1 - y_n'(0)\|_{L^2}^2\right\} =$$

$$= 16\{\|\beta(u_1)\|_{L^2}^2 + K_\beta^2 \|u_1 - y_n'(0)\|_{L^2}^2\} + K_3^2,$$

being $K_3 = 2\left\{\|u_1\|_{H_0^1}^2 + \left(\int_0^T \|f'(\eta)\|_{L^2}\, d\eta\right)^2 + 4\|Au_0\|_{L^2}^2 + 4\|f(0)\|_{L^2}^2\right\}^{1/2}$ independent of $\beta(\eta)$ and n.

d) Denoting by K_1, K_2, K_3 quantities which do not depend on $\beta(\eta)$ and n, the following relations therefore hold $\forall t \in 0|\!\!-\!\!|T$:

$$(3.26)\quad \begin{aligned} &\|y_n(t)\|_{H_0^1} \leq K_1 \quad , \quad \|y_n'(t)\|_{H_0^1} \leq K_2, \\ &\|Ay_n(t)\|_{L^2} \leq K_2 \quad , \quad \|y_n''(t)\|_{L^2} \leq 4\{\|\beta(u_1)\|_{L^2} + K_\beta \|y_n'(0) - u_1\|_{L^2}\} + K_3. \end{aligned}$$

As the sequence $\{Ay_n(t)\}$ is L^2–bounded, it follows that $\{y_n(t)\}$ is H_0^1–relatively compact $\forall t \in 0 \vert\!-\!\vert T$; being moreover $\| y_n'(t) \|_{H_0^1} \leq K_2$, the functions $y_n(t)$ are H_0^1–equi-continuous. Hence, by the (vectorial) theorem of Ascoli-Arzelà, we can assume that it results

$$(3.27) \qquad \lim_{n \to \infty} y_n(t) \underset{H_0^1}{=} y(t) \quad \text{uniformly on} \quad 0 \vert\!-\!\vert T .$$

In the same way it follows, from the second and fourth of (3.26) (being the embedding of H_0^1 in L^2 completely continuous) that

$$(3.28) \qquad \lim_{n \to \infty} y_n'(t) \underset{L^2}{=} y'(t) \quad \text{uniformly on} \quad 0 \vert\!-\!\vert T .$$

The limit function $y(t)$ is therefore, by (3.27), (3.28), E–continuous on $0 \vert\!-\!\vert T$.

By (3.26), it is moreover possible to assume that it results

$$(3.29) \qquad \begin{aligned} &\lim_{n \to \infty}{}^{*} y_n'(t) \underset{L^2(0 \vert-\vert T;\, H_0^1)}{=} y'(t) , \\ &\lim_{n \to \infty}{}^{*} y_n''(t) \underset{L^2(0 \vert-\vert T;\, L^2)}{=} y''(t) , \\ &\lim_{n \to \infty}{}^{*} Ay_n(t) \underset{L^2(0 \vert-\vert T;\, L^2)}{=} Ay(t) , \end{aligned}$$

where lim* denotes the limit in the weak topology.

We recall now that, if a sequence $\{z_n(t)\}$ is such that $z_n(t) \in L^2(0 \vert\!-\!\vert T; X)$ (X Hilbert space), $\|z_n(t)\|_X \leq K$, $\lim\limits_{n \to \infty}{}^{*} z_n(t) \underset{L^2(0 \vert-\vert T;\, L^2)}{=} z(t)$, then $\| z(t) \|_X \leq K$ almost everywhere.

The limit function $y(t)$ therefore satisfies, by (3.26) and (3.29), the relations

$$(3.30) \qquad \begin{aligned} &\| y(t) \|_{H_0^1} \leq K_1 \quad , \quad \| y'(t) \|_{H_0^1} \leq K_2 , \\ &\| Ay(t) \|_{L^2} \leq K_2 \quad , \quad \| y''(t) \|_{L^2} \leq 4 \, \| \beta(u_1) \|_{L^2} + K_3 , \end{aligned}$$

where K_1, K_2, K_3 do not depend on $\beta(\eta)$. Observe that $\| y''(t) \|_{L^2}$ has been estimated independently of K_β, while this did not occur for $\| y_n''(t) \|_{L^2}$.

As $y(t) \in C^0(0 \vert\!-\!\vert T; E)$, it follows from (3.30) that $y(t) \in \Gamma$.

By the Fischer-Riesz theorem, we can assume, by (3.28), that it results, almost everywhere on Q,

$$(3.31) \qquad \lim_{n \to \infty} \frac{\partial y_n(t, x)}{\partial t} = \frac{\partial y(t, x)}{\partial t}$$

and consequently ($\beta(\eta)$ being continuous),

$$(3.32) \qquad \lim_{n \to \infty} \beta\left(\frac{\partial y_n(t, x)}{\partial t} \right) = \beta\left(\frac{\partial y(t, x)}{\partial t} \right).$$

As $\beta(\eta)$ is bounded, it follows, $\forall \zeta \in L^2(Q)$,

$$\lim_{n\to\infty} \int_Q \beta\left(\frac{\partial y_n(t,x)}{\partial t}\right) \zeta(t,x)\,dx\,dt = \int_Q \beta\left(\frac{\partial y(t,x)}{\partial t}\right) \zeta(t,x)\,dx\,dt. \tag{3.33}$$

In particular, if $\zeta(t,x) = g_j(x)\,\varphi(t)$ $(\varphi(t) \in C^0(0 \vert\!-\!\vert T))$, we obtain

$$\lim_{n\to\infty} \int_0^T (\beta(y_n'(t)), g_j)_{L^2}\, \varphi(t)\,dt = \int_0^T (\beta(y'(t)), g_j)_{L^2}\, \varphi(t)\,dt. \tag{3.34}$$

From (3.27), (3.28), (3.29), (3.34) it follows then

$$\lim_{n\to\infty} \int_0^T \{(y_n''(t), g_j)_{L^2} - (Ay_n(t), g_j)_{L^2} + (\beta(y_n'(t)), g_j)_{L^2} - (f(t), g_j)_{L^2}\}\, \varphi(t)\,dt = \tag{3.35}$$

$$= \int_0^T \{(y''(t), g_j)_{L^2} - (Ay(t), g_j)_{L^2} + (\beta(y'(t)), g_j)_{L^2} - (f(t), g_j)_{L^2}\}\, \varphi(t)\,dt$$

and, by (3.2),

$$y''(t) - Ay(t) + \beta(y'(t)) - f(t) = 0. \tag{3.36}$$

The function $y(t) \in \Gamma$ therefore satisfies the given equation (1.9). As it verifies obviously the initial conditions (1.14), the theorem is proved.

We observe, finally, that, by (3.30), (3.36) it results

$$\|\beta(y'(t)\|_{L^2} \leq 4\,\|\beta(u_1)\|_{L^2} + K_4, \tag{3.37}$$

K_4 being independent of $\beta(\eta)$.

Let us now prove the final existence theorem.

Let us, at first, construct a suitable sequence $\{\beta_n(\eta)\}$ of functions continuous and bounded on R^1 together with their first derivatives, with $\beta_n'(\eta) \geq 0$.

Let $\{\rho_n\}$, $\{\sigma_n\}$ be two sequences such that $0 < \rho_n < \rho_{n+1} \to b$, $0 > \sigma_n > \sigma_{n+1} \to a$; assume, moreover, that $\beta(\eta)$ is continuous in ρ_n and σ_n, $\forall n$. Bearing in mind that $\beta(0^+) \geq 0$, $\beta(0^-) \leq 0$, we set, for $n = 1, 2, \cdots$,

$$\tag{3.38} \begin{aligned} \beta_n^+(\eta) &= \begin{cases} \beta(\eta) & \text{when } 0 < \eta < \rho_n \\ \beta(\rho_n) & \text{»} \quad \rho_n \leq \eta < +\infty \\ 0 & \text{»} \quad \eta \leq 0 \end{cases} \\ \beta_n^-(\eta) &= \begin{cases} 0 & \text{when } \eta \geq 0 \\ \beta(\eta) & \text{»} \quad 0 > \eta > \sigma_n \\ \beta(\sigma_n) & \text{»} \quad -\infty < \eta \leq \sigma_n . \end{cases} \end{aligned}$$

Let $\varphi(\eta)$ be a function $\in C^\infty(R^1)$ such that

$$(3.39)\qquad \varphi(\eta) \geq 0 \quad , \quad \int_{R^1} \varphi(\eta)\, d\eta = 1, \qquad \text{supp}\, \varphi(\eta) \subset \overline{0\ \vartheta}.$$

where $\vartheta = \min(\rho_1, -\sigma_1)$.

Setting

$$(3.40)\qquad \tilde{\beta}_n^+(\eta) = n \int_{R^1} \beta_n^+(\eta - \tau)\, \varphi(n\tau)\, d\tau = n \int_0^{\vartheta/n} \beta_n^+(\eta - \tau)\, \varphi(n\tau)\, d\tau =$$
$$= n \int_{R^1} \beta_n^+(\tau)\, \varphi(n(\eta - \tau))\, d\tau,$$
$$\tilde{\beta}_n^-(\eta) = n \int_{R^1} \beta_n^-(\eta + \tau)\, \varphi(n\tau)\, d\tau = n \int_0^{\vartheta/n} \beta_n^-(\eta + \tau)\, \varphi(n\tau)\, d\tau =$$
$$= n \int_{R^1} \beta_n^-(\tau)\, \varphi(n(\tau - \eta))\, d\tau,$$

it results $\tilde{\beta}_n^+(\eta)$, $\tilde{\beta}_n^-(\eta) \in C^\infty(R^1)$. Moreover $\tilde{\beta}_n^+(\eta)$, $\tilde{\beta}_n^-(\eta)$ are non-decreasing and it results

$$(3.41)\qquad \tilde{\beta}_n^+(\eta) = 0 \quad \text{when } \eta \leq 0 \quad , \quad \tilde{\beta}_n^-(\eta) = 0 \quad \text{when } \eta \geq 0.$$

Setting

$$(3.42)\qquad \beta_n(\eta) = \tilde{\beta}_n^+(\eta) + \tilde{\beta}_n^-(\eta),$$

it is then, by (1.15),

$$(3.43)\qquad \beta_n(0) = 0 \quad , \quad 0 \leq \beta_n(\eta) \leq \beta(\eta^-) = \bar{\beta}(\eta) \qquad \text{when } \eta \geq 0,$$
$$\bar{\beta}(\eta) = \beta(\eta^+) \leq \beta_n(\eta) \leq 0 \qquad \text{when } \eta \leq 0.$$

Observe that $\beta_n(\eta)$ is a non-decreasing function, bounded on R^1 together with all its derivatives. Moreover, the following property holds:

Let us fix $\eta \in \overline{a\ b}$. Chosen ε arbitrarily, there exist δ_ε and n_ε (depending also on η) such that, when $|\xi - \eta| < \delta_\varepsilon$, $n > n_\varepsilon$, it results

$$(3.44)\qquad \beta(\eta^-) - \varepsilon < \beta_n(\xi) < \beta(\eta^+) + \varepsilon.$$

Assume, at first, $0 < \eta < b$. We take δ'_ε with $0 < \delta'_\varepsilon < \eta$, $\eta + \delta'_\varepsilon < b$, such that, when $|\xi - \eta| < \delta'_\varepsilon$, it is

$$(3.45)\qquad \beta(\eta^-) - \varepsilon < \beta(\xi) < \beta(\eta^+) + \varepsilon.$$

We choose then $\delta_\varepsilon > 0$ and n_ε such that $\delta_\varepsilon + \dfrac{\vartheta}{n_\varepsilon} < \delta'_\varepsilon$, $\eta + \delta_\varepsilon < \rho_{n_\varepsilon}$. It results, when $n > n_\varepsilon$, $|\xi - \eta| < \delta_\varepsilon$ ($\Rightarrow \xi < \rho_{n_\varepsilon}$),

$$(3.46)\qquad \beta_n(\xi) = n \int_0^{\vartheta/n} \beta_n^+(\xi - \tau)\, \varphi(n\tau)\, d\tau = n \int_0^{\vartheta/n} \beta(\xi - \tau)\, \varphi(n\tau)\, d\tau.$$

Being

$$\eta - \delta'_\varepsilon < \eta - \delta_\varepsilon - \frac{\vartheta}{n} < \xi - \tau < \eta + \delta_\varepsilon < \eta + \delta'_\varepsilon, \tag{3.47}$$

from (3.46) follows (3.44).

In the same way it is possible to prove (3.44) when $\eta < 0$.

Let, finally, be $\eta = 0$. It is then $\beta_n(0) = 0$ and therefore, when $\xi > 0$,

$$0 \leq \beta_n(\xi) = n \int_0^{\vartheta/n} \beta_n^+(\xi - \tau)\, \varphi(n\tau)\, d\tau \leq \beta(\xi^-). \tag{3.48}$$

Hence, if $0 \leq \xi < \delta_\varepsilon$, it results

$$0 \leq \beta_n(\xi) < \beta(0^+) + \varepsilon. \tag{3.49}$$

In the same way it can be proved that, when $-\delta_\varepsilon < \xi \leq 0$,

$$\beta(0^-) - \varepsilon < \beta_n(\xi) \leq 0. \tag{3.50}$$

Hence (3.44) holds also when $\eta = 0$.

Let us now prove the existence theorem.

Consider the sequence $\{\beta_n(\eta)\}$ defined above and let $y_n(t)$ be the solution, in Γ, of the equation

$$A y_n(t) - y''_n(t) + f(t) = \beta_n(y'_n(t)) \tag{3.51}$$

satisfying the initial conditions

$$y_n(0) = u_0 \quad , \quad y'_n(0) = u_1 . \tag{3.52}$$

By theorems I and II such a solution exists and is unique.

It results then, by (3.30), (3.37), being, by construction, $\|\beta_n(u_1)\|_{L^2} \leq \leq \|\bar{\beta}(u_1)\|_{L^2}$:

$$\begin{gathered} \|y_n(t)\|_{H_0^1} \leq K_1 \quad , \quad \|y'_n(t)\|_{H_0^1} \leq K_2 , \\ \|A y_n(t)\|_{L^2} \leq K_2 \quad , \quad \|y''_n(t)\|_{L^2} \leq 4\|\beta_n(u_1)\|_{L^2} + K_3 \leq 4\|\bar{\beta}(u_1)\|_{L^2} + K_3 , \\ \|\beta_n(y'_n(t))\|_{L^2} \leq 4\,\|\beta_n(u_1)\|_{L^2} + K_4 \leq 4\,\|\bar{\beta}(u_1)\|_{L^2} + K_4 , \end{gathered} \tag{3.53}$$

where the quantities K_j do not depend on n.

By (3.53) we can assume, as before, that it results

$$\lim_{n\to\infty} y_n(t) \underset{E}{=} y(t) \qquad \text{uniformly on } 0 \vdash\!\dashv T, \tag{3.54}$$

$$\begin{gathered} \lim_{n\to\infty}{}^{*}\, y'_n(t) \underset{L^2(0 \vdash\!\dashv T;\, H_0^1)}{=} y'(t), \\ \lim_{n\to\infty}{}^{*}\, y''_n(t) \underset{L^2(0 \vdash\!\dashv T;\, L^2)}{=} y''(t), \\ \lim_{n\to\infty}{}^{*}\, A y_n(t) \underset{L^2(0 \vdash\!\dashv T;\, L^2)}{=} A y(t). \end{gathered} \tag{3.55}$$

We can, moreover, assume that

$$\lim_{n\to\infty} \frac{\partial y_n(t,x)}{\partial t} = \frac{\partial y(t,x)}{\partial t} \tag{3.56}$$

almost everywhere on Q and, by the last of (3.53),

$$\operatorname*{lim^*}_{n\to\infty} \beta_n\left(\frac{\partial y_n(t,x)}{\partial t}\right) \underset{L^2(Q)}{=} \chi(t,x). \tag{3.57}$$

Therefore $y(t) \in C^0(0 \vdash\!\dashv T; E)$ and

$$\|y'(t)\|_{H_0^1} \leq K_2 \quad , \quad \|Ay(t)\|_{L^2} \leq K_2, \tag{3.58}$$

$$\|y''(t)\|_{L^2} \leq 4\,\|\bar\beta(u_1)\|_{L^2} + K_3 .$$

Hence $y(t) \in \Gamma$.

By (3.57) and a theorem of Mazur there exists, $\forall n$, a sequence $\{\rho_{nk};\ k \geq n\}$, with $\rho_{nk} \geq 0$, $\sum_1^\infty{}_k \rho_{nk} = 1$, such that

$$\lim_{n\to\infty} \sum_n^\infty{}_k \rho_{nk}\,\beta_k\left(\frac{\partial y_k(t,x)}{\partial t}\right) \underset{L^2(Q)}{=} \chi(t,x). \tag{3.59}$$

Hence it is possible to extract a subsequence $\{n_j\} \subseteq \{n\}$ such that, almost everywhere on Q,

$$\lim_{j\to\infty} \sum_{n_j}^\infty{}_k \rho_{n_j k}\,\beta_k\left(\frac{\partial y_k(t,x)}{\partial t}\right) = \chi(t,x). \tag{3.60}$$

Let $(\bar t, \bar x)$ be a point in which (3.56) and (3.60) hold. Setting $\bar\eta = y_t(\bar t, \bar x)$ and assuming $a < \bar\eta < b$, we shall prove that

$$\beta(\bar\eta^-) \leq \chi(\bar t, \bar x) \leq \beta(\bar\eta^+). \tag{3.61}$$

Fixed an arbitrary $\varepsilon > 0$, let us determine δ_ε and n'_ε in such a way that, when $|\xi - \bar\eta| < \delta_\varepsilon$, $n > n'_\varepsilon$, it results

$$\beta(\bar\eta^-) - \varepsilon < \beta_n(\xi) < \beta(\bar\eta^+) + \varepsilon .$$

We now choose $n_\varepsilon > n'_\varepsilon$ such that, when $n > n_\varepsilon$, it results

$$\left| \frac{\partial y_n(\bar t, \bar x)}{\partial t} - \frac{\partial y(\bar t, \bar x)}{\partial t} \right| < \delta_\varepsilon .$$

It is then

$$\beta(\bar\eta^-) - \varepsilon < \beta_n\left(\frac{\partial y_n(\bar t, \bar x)}{\partial t}\right) < \beta(\bar\eta^+) + \varepsilon,$$

that is, always for $n > n_\varepsilon$,

$$\beta(\bar\eta^-) - \varepsilon < \sum_n^\infty{}_k \rho_{nk}\,\beta_k\left(\frac{\partial y_k(\bar t, \bar x)}{\partial t}\right) < \beta(\bar\eta^+) + \varepsilon .$$

Hence, by (3.60),

$$\beta(\overline{\eta}^-) - \varepsilon \leq \chi(\bar{t}, \bar{x}) \leq \beta(\overline{\eta}^+) + \varepsilon$$

which, being ε arbitrary, proves (3.61).

Consider, finally, the case when $b < +\infty$, which implies $\lim\limits_{\eta \to b^-} \beta(\eta) = +\infty$. Let Q_b be the set where $y_t(t, x) \geq b$: it cannot be $m(Q_b) > 0$. In fact, if it were so, there would exist, by (3.56), a set $Q'_b \subseteq Q_b$, with $m(Q'_b) > 0$, such that (3.56) holds uniformly on Q'_b. Since $y_t(t, x) \geq b$, we can take, $\forall n, p_n \geq n$ such that

$$\frac{\partial y_{p_n}(t, x)}{\partial t} \geq \rho_n \qquad \forall (t, x) \in Q'_b .$$

Hence

$$\int\limits_{Q'_b} \beta_{p_n}\left(\frac{\partial y_{p_n}(t, x)}{\partial t}\right) dx\, dt \geq \tilde{\beta}_n(\rho_n)\, m(Q'_b) \to +\infty ,$$

against the last of (3.53).

From (3.51), (3.54), (3.55), (3.61) it follows that $y(t)$ satisfies the equation

$$Ay(t) - y''(t) + f(t) = \beta(y'(t))$$

in the sense indicated in § 1. Since $y(t) \in \Gamma$ and the initial conditions (1.14) are, by (3.52), (3.54), obviously verified, the existence theorem is proved.