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## The Riemannian structure of space-time as a consequence of physical hypotheses

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**Fisica matematica.** — *The Riemannian structure of space-time as a consequence of physical hypotheses.* Nota di MARIO CASTAGNINO (*), presentata (**) dal Corrisp. C. CATTANEO.

RIASSUNTO. — Si dimostra che dalla ipotesi che lo spazio–tempo $V_4$ sia una varietà differenziabile dotata di una connessione lineare, dal principio fisico di equivalenza e dall'ipotesi che la curvatura spazio–temporale non abbia influenza sulle costanti universali $c$ ed $h$, discende come conseguenza necessaria il carattere riemanniano di $V_4$.

## 1. INTRODUCTION.

General Relativity is based on the *Riemannian Hypothesis*, which establishes that the space-time is endowed with a mathematical structure of Riemannian Manifold. It is either introduced as a postulate or it is explained as a consequence of certain geometrical hypotheses, that usually consist in:

*a*) the assumption that local inertial frames, where gravitational forces vanish, are small carthesian triads in " uniform motion ";

*b*) the existence of parallelograms, in local, approximately flat, space.

But any one of these conjectures has a geometric character and they can hardly be proved directly, even by an ideal experience; so we think that a system of physical hypotheses yielding to the same goal is useful, even more, if the new hypotheses are better based, are more precisely stated and allow us more rigorous deductions.

Obviously we are forced to make some basic geometric assumptions to define our geometrical background:

H. 1. *The space-time has a structure of differentiable manifold*: $V_4$.

H. 2. *This manifold is connected by a linear connection* $\Gamma^i_{jk}$.

We shall study our problem only in connection with a set of " classical " phenomena: inertial-gravitational and electromagnetic ones. So we are only interested in the motion of free charged particles in (inertial-) gravitational and electromagnetic fields. (We make one exception: in our research we also include the motion of a free photon in a (inertial-) gravitational field). About this kind of phenomena we set a fundamental physical hypothesis, the *Principle of Equivalence*:

H. 3. *For every point of space-time there always exists a coordinate system—we shall call it a local-inertial frame—in which gravitation has no influence either on the motion of particles or on any other physical process.*

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*The laws of Special Relativity, expressed in the local inertial frame, are valid in that point* (1).

Thus, in a particular point, all the tensorial entities that exist in Special Relativity exist too in a local inertial frame, and therefore in all the frames of General Relativity. And all the algebraic laws that vinculate these tensorial entities in Special Relativity, are valid too in all the frames of General Relativity. In particular we arrive to an important conclusion: the metric tensor $g_{ij}$ exists and it keeps its usual properties.

Let us now consider the main problem: Which is the connection of the manifold?

We shall find such a connection and prove that it is not unique. Nevertheless the ambiguity is physically unimportant, for our set of phenomena. Based on the fact that the space-time curvature has no influence on the constancy of the universal constants $c$ and $h$ we shall demonstrate that this connection is the Riemannian one.

## 2. Mathematical preliminaries and usual hypotheses.

Let us consider a four-dimensional differentiable manifold $V_4$: In every point of $V_4$, where we call $\tau$ the tangent space and, as we have just said based on H. 3, the metric tensor $g_{ij} \in \tau \otimes \tau$ exists (signature $+, +, +, -$).

(1) The " strong " principle of equivalence, as stated by *Pauli* (cf. [13]) says: " For every infinitely small world region (i.e. a world region which is so small that the space-time variation of gravity can be neglected in it) there always exists a coordinate system $K_0$ ($X_1$, $X_2$, $X_3$, $X_4$) in which gravitation has no influence either on the motion of particles or any other physical process ".

H. 3 is a little different, we have changed the small region by the tangent space, in order to avoid the problem of the region's size (cf. [6], [7] and [8]).

We must also establish a criterion to know how to choose between two different laws, of Special Relativity, regarding to a physical phenomena, $\left(e.g.\ (a)\ \frac{du^i}{d\tau} = 0,\ (b)\ \frac{d^2 u^i}{d\tau^2} = 0\right)$ which will lead us to contradictory laws in General Relativity. We shall always choose the law that yields to a better approximation between Special Relativity and General Relativity, in the local inertial frame (e. g. Law $(a)$ leads to:

$$x^i = x_0^i + u^i \tau + O(\tau^2),$$

law $(b)$ conduces to

$$x^i = x_0^i + u^i \tau + \frac{1}{2} \frac{du^i}{d\tau} \tau^2 + O(\tau^3),$$

the laws of Special Relativity both yield to:

$$x^i = x_0^i + u^i \tau,$$

so law $(a)$ is the right one. A less trivial example: in electromagnetic theory we must transform the laws $\partial_i F^{ik} = J^k$ and $\partial_i \overset{*}{F}{}^{ik} = 0$ and not the law $\Delta A^i = J^k$, i. e. the law that contains the connection coefficient and not its derivatives). Of course this criterion coincides with the usual choice, and can be regarded as a more precise statement about the size of the region.

We shall call $g^{ij}$ the inverse matrix of $g_{ij}$, $\Gamma^i_{jk}$ the coefficients of the most general linear connection and $\left\{ {i \atop jk} \right\}$ those of the Riemannian connection. The difference

$$\Gamma^i_{jk} - \left\{ {i \atop jk} \right\} \equiv T_{jk}{}^i + S_{jk}{}^i, \tag{1}$$

is a tensor; $T_{jk}{}^i$ is the symmetric part of such a tensor and

$$S_{jk}{}^i \equiv \frac{1}{2} (\Gamma^i_{jk} - \Gamma^i_{kj}), \tag{2}$$

is the antisymmetric part, i.e. the torsion. The derivative of the metric tensor, with respect to the linear connection $\Gamma^i_{jk}$, is:

$$\nabla_i g_{jk} \equiv \partial_i g_{jk} - \Gamma^h_{ji} g_{hk} - \Gamma^h_{ki} g_{jh} . \tag{3}$$

By a rotation of indices we obtain two equivalent equations. If we add the first equation to the second and subtract the third one we get (cfr. [14]):

$$\Gamma^k_{ij} \equiv \left\{ {k \atop ij} \right\} - \begin{bmatrix} k \\ ij \end{bmatrix} + S_i{}^k{}_j + S_j{}^k{}_i + S_{ij}{}^k, \tag{4}$$

where $\begin{bmatrix} k \\ ij \end{bmatrix}$ is a "Christoffel symbol" written with covariant derivatives:

$$\begin{bmatrix} k \\ ij \end{bmatrix} \equiv \frac{1}{2} g^{kh} (\nabla_i g_{jh} + \nabla_j g_{ih} - \nabla_h g_{ij}). \tag{5}$$

So we have:

$$T_{ij}{}^k \equiv - \begin{bmatrix} k \\ ij \end{bmatrix} + S_i{}^k{}_j + S_j{}^k{}_i . \tag{6}$$

We shall prove that the connection is a Riemannian one; if we reach to:

$$T_{ij}{}^k = 0 \quad ; \quad S_{ij}{}^k = 0. \tag{7}$$

The geometric assumptions usually stated are suggested by the Equivalence Principle. In fact, it is assumed that in a neighborhood of a given point, where we can consider space-time as approximately flat, everything happens as in Special Relativity. Hence inertial frames are small carthesian triads in "uniform motion", so the space-time path of a free particle is locally a straight line; and the metric tensor, in the inertial frame, is constant. It follows that this path is a geodesic and that $T_{ijk} = 0$; furthermore the existence of parallelograms, in the locally flat space, assure that the torsion vanishes, the connection is then a Riemannian one. But, as we have said, all these conjectures, and also some others that are frequently used, have a geometric character, they are stated in an unprecise way, and do not lead us to rigorous deductions and can be substituted by a set of more precise physical hypotheses as follows.

## 3. First consequence of the Equivalence Principle. The law of motion of a free particle.

Let us return to H. 3. The Principle of Equivalence, by itself only, allows us to find the law of motion of a free particle. In fact, in the local inertial frame $S^0$ the law of motion of such a particle is:

$$\frac{dU^{i^0}}{d\tau} = 0, \tag{8}$$

where the vector $U^{i^0}$ is the absolute velocity and $\tau$ the proper time. In an arbitrary frame S the absolute velocity is:

$$U^i = A^i_{i^0} U^{i^0}, \tag{9}$$

so in that system the law of motion is:

$$\frac{dU^i}{d\tau} + (A^i_{i^0} \partial_k A^{i^0}_j) U^k U^j = 0, \tag{10}$$

but we know that:

$$A^i_{i^0} \partial_k A^{i^0}_j = \left\{ {i \atop kj} \right\} - A^i_{i^0} A^{j^0}_j A^{k^0}_k \left\{ {i^0 \atop j^0 k^0} \right\}. \tag{11}$$

Substituting in (11) we get:

$$\frac{dU^i}{d\tau} + \left[ \left\{ {i \atop jk} \right\} - A^i_{i^0} A^{j^0}_j A^{k^0}_k \left\{ {i^0 \atop j^0 k^0} \right\} \right] U^k U^j = 0 . \tag{12}$$

But we realize that $A^i_{i^0} A^{j^0}_j A^{k^0}_k \left\{ {i^0 \atop j^0 k^0} \right\}$ is a symmetric tensor in $j , k$ so we can define a linear connection:

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + T_{jk}{}^i + S_{jk}{}^i, \tag{13}$$

where [2]

$$T_{jk}{}^i = - A^i_{i^0} A^{j^0}_j A^{k^0}_k \left\{ {i^0 \atop j^0 k^0} \right\}, \tag{14}$$

and where $S_{jk}{}^i$ is an arbitrary torsion. From now on we shall study this particular connection, i.e. the connection $\Gamma^i_{jk}$ with the important property that the law of motion of a free particle (12) can be written:

$$\frac{DU^i}{d\tau} = \frac{dU^i}{d\tau} + \Gamma^i_{jk} U^j U^k = 0. \tag{15}$$

(2) Incidentally, the usual hypothesis $\partial_{i^0} g_{j^0 k^0} = 0$ yields to $\left\{ {i^0 \atop j^0 k^0} \right\} = 0$, i. e. $T_{jk}{}^i = 0$.

We have proved that the law of motion of a free particle, in the connection $\Gamma^i_{jk}$ is, (15). Thus the inertial frame is characterized for the vanishing of the symmetric part of the coefficients of the connection:

$$\Gamma^i_{(jk)} = \left\{ {i \atop j\,k} \right\} + \mathrm{T}_{jk}{}^i = 0, \tag{16}$$

i.e. the vanish of the inertial-gravitational " forces ".

## 4. THE " SCHRÖDINGER HYPOTHESIS ".

We know that in Special Relativity the norm of the absolute velocity of a particle is a constant:

$$g_{ij} \mathrm{U}^i \mathrm{U}^j = -c^2 = \text{const.} \quad ; \quad \mathrm{U}^i \equiv \frac{\mathrm{d}x^i}{\mathrm{d}\tau} . \tag{17}$$

Let us suppose that this is also the case in General Relativity and let us postulate:

H. 4. *In General Relativity the law of motion of a free particle* (15) *agrees with* (17).

Deriving (17), with respect to $\Gamma^i_{jk}$, and having in mind (15) we have:

$$\left(\frac{\mathrm{D}}{\mathrm{d}\tau} g_{ij}\right) \mathrm{U}^i \mathrm{U}^j = 0, \tag{18}$$

i.e.

$$\mathrm{U}^k (\nabla_k g_{ij}) \mathrm{U}^i \mathrm{U}^j = 0, \tag{19}$$

where $\mathrm{U}^i$ has an arbitrary time-like direction so the metric tensor must satisfy the equations:

$$\nabla_i g_{jk} + \nabla_j g_{ki} + \nabla_k g_{ij} = 0 . \tag{20}$$

From (6) we can deduce:

$$\nabla_i g_{jk} + \nabla_j g_{ki} - \nabla_k g_{ij} \equiv 2 g_{kh} \begin{bmatrix} h \\ ij \end{bmatrix} . \tag{21}$$

Subtracting (21) from (20) we get:

$$\nabla_k g_{ij} = - g_{kh} \begin{bmatrix} h \\ ij \end{bmatrix} , \tag{22}$$

hence, by (7) we can write $\mathrm{T}_{jk}{}^i$ as:

$$\mathrm{T}_{ijk} = \nabla_k g_{ij} + \mathrm{S}_{ikj} + \mathrm{S}_{jki} . \tag{23}$$

Taking again into account (20) and the antisymmetry of $\mathrm{S}_{jk}{}^i$ it results:

$$\mathrm{T}_{ijk} + \mathrm{T}_{jki} + \mathrm{T}_{kij} = 0 . \tag{24}$$

On the contrary (24) implies (20).

The equation (24) was introduced by *Schrödinger*, making use of different arguments, so we can call H. 4 the *Schrödinger Hypothesis* (cfr. [15]). All connections where H. 4 is valid have the following geometric property: " If a vector is parallely transferred along a curve, in such a way that it is always tangent to the curve, it has a constant norm " (3).

The hypothesis has also a clear physical meaning: if (17) is valid, along the space-time path of a particle, its proper time is:

$$d\tau^2 = -\frac{1}{c^2} g_{ij}\, dx^i\, dx^j \,, \tag{25}$$

i.e. (17) is the chrometric hypothesis of proper time (cfr. [12] and [16]): "A clock that follows the motion of the test particle marks a $d\tau$ proportional to $\frac{1}{c}\sqrt{-g_{ij}\, dx^i\, dx^j}$ , that is to say, proportional to the world-interval ".

## 5. THE CONSTANCY OF PLANCK'S $h$.

It is known that *Planck's h* is considered a constant in the theories where the space-time has a structure of Riemannian Manifold. (Cfr. [17]) (4). Let us now take this fact as a postulate:

H. 5. *The connection must be such a one that we can consider the h of Planck as a constant.*

In special Relativity the linear 4–momentum of a photon is:

$$P^i = \frac{h\nu}{c^2}\frac{dx^i}{dt}, \tag{26}$$

where $t$ is the coordinate time, $x^4 = ct$, and its law of motion is:

$$\frac{dP^i}{dt} = 0. \tag{27}$$

In General Relativity, if we adopt a physically admissible frame of reference defined by a family of time-like curves, with tangent vector $\gamma^i$, and we introduce the relative standard time (cfr. [2]):

$$dT = -\frac{1}{c} g_{ij}\gamma^i\, dx^j, \tag{28}$$

the linear momentum of the photon is (cfr. [3]):

$$P^i = \frac{h\nu}{c^2}\frac{dx^i}{dT}. \tag{29}$$

(3) Of course this curve is an " affine " geodesic i.e. wit a parallely transported tangent vector. The equation commonly accepted in place of (20): $\nabla_i g_{jk} = 0$ leads to the stronger property " Vectors which are parallely transferred have a constant norm ".

(4) *L. Infeld* reach to a different conclusion, but his arguments are based in the choice of a coordinate time, instead of the standard physical time. (cf. [9] and [10]).

We can find the law of motion of the photon repeating the arguments that lead us from equation (8) to equation (15). In this way if we start from (27), written under the form:

$$\frac{dP^i}{dT} = 0, \tag{27'}$$

using H. 3 we reach to:

$$\frac{DP^i}{dT} = 0. \tag{30}$$

The world-path of a photon is then a null geodetic. This curve admits an affine parameter $u$, characterized by the following property:

$$\frac{DU^i}{du} = 0, \tag{31}$$

$U^i$ being the tangent vector $\frac{dx^i}{du}$.

From (29) and (30) we have:

$$\frac{DP^i}{dT} = \frac{d}{dT}\left(\frac{h\nu}{c^2}\right)\frac{dx^i}{dT} + \left(\frac{h\nu}{c^2}\right)\frac{D}{dT}\left(\frac{dx^i}{dT}\right) = 0, \tag{32}$$

but

$$\frac{D}{dT}\left(\frac{dx^i}{dT}\right) = \frac{D}{dT}\left(\frac{dx^i}{du}\cdot\frac{du}{dT}\right) = -\frac{dx^i}{dT}\,\frac{\frac{d}{dT}\left(\frac{dT}{du}\right)}{\left(\frac{dT}{du}\right)},$$

where

$$\frac{dT}{du} = -\frac{1}{c}\,g_{ij}\,\gamma^i\,\frac{dx^j}{du} = -\frac{1}{c}\,g_{ij}\,\gamma^i\,U^j.$$

Replacing the last equation in (32) we have:

$$\frac{\frac{d}{dT}\left(\frac{h\nu}{c^2}\right)}{\left(\frac{h\nu}{c^2}\right)} - \frac{\frac{d}{dT}\left(\frac{dT}{du}\right)}{\left(\frac{dT}{du}\right)} = 0,$$

i.e.

$$h\nu = K\,g_{ij}\,\gamma^i\,U^j, \tag{33}$$

where K is a constant.

On the other hand we can compute the variation of the frequency, in different points of the manifold, measured in the frame $\gamma^i$. Let us consider a set of $\infty^3$ photons whose space-time paths constitute a congruence of curves:

$$x^i = x^i(u, v, v', v''), \tag{34}$$

where $u$ is the affine parameter of each curve and $v$, $v'$ and $v''$ are three parameters which define each curve. Let us choose these parameters in such a way that calling:

$$U^i \equiv \frac{\partial x^i}{\partial u} \quad ; \quad V^i \equiv \frac{\partial x^i}{\partial v}, \tag{35}$$

$V^i$ would be parallel to $\gamma^i$, in every point. Let us call

$$V = (-g_{ij} V^i V^j)^{1/2}, \tag{36}$$

so V is the standard time, measured in the frame $\gamma^i$, between the passages of two rays separated by a difference $\Delta v = 1$, of the parameter $v$; hence the frequency is proportional to the inverse of V:

$$\nu = \frac{H}{V}, \tag{37}$$

where H is a constant. From (36) and (37) we have:

$$g_{ij} V \gamma^i U^j = \frac{H}{K} h, \tag{38}$$

and by (36):

$$g_{ij} V^i U^j = \frac{H}{K} h. \tag{39}$$

We have postulate that $h$ is a constant, thus:

$$\frac{d}{du}(g_{ij} V^i U^j) = 0, \tag{40}$$

so:

$$\left(\frac{D}{du} g_{ij}\right) V^i U^j + g_{ij} \frac{DV^i}{du} U^j = 0. \tag{41}$$

But from (34) we have:

$$\frac{DV^i}{du} = \frac{DU^i}{dv} + 2 S_{jk}{}^i V^j U^k, \tag{42}$$

but $g_{ij} U^i U^j = 0$, so we have:

$$\left(\frac{D}{dv} g_{ij}\right) U^i U^j + 2 g_{ij} U^i \frac{DU^j}{dv} = 0, \tag{43}$$

hence from (41), (42) and (43):

$$g_{ij} \frac{DV^i}{du} U^j = -\frac{1}{2}\left(\frac{D}{dv} g_{ij}\right) U^i U^j + 2 S_{hkj} V^h U^k U^j = 0, \tag{44}$$

substituting the last one in (41) we have:

$$\left(\nabla_i g_{jk} - \frac{1}{2} \nabla_j g_{ik} + 2 S_{jik}\right) V^j U^i U^k = 0. \tag{45}$$

Thus $V^j$ being arbitrary, it follows:

$$\left[\frac{1}{2}(\nabla_i g_{jk} + \nabla_k g_{ji} - \nabla_j g_{ik}) + S_{jik} + S_{jki}\right] U^i U^k = 0. \tag{46}$$

Having in mind (6) and (5) we have:

$$T_{ikj}\,U^i U^k = 0. \tag{47}$$

But $U^i$ is an arbitrary null vector:

$$g_{ik}\,U^i U^k = 0, \tag{48}$$

so the hypothesis $h =$ const. yields [(5)]:

$$T_{ikj} = \lambda_j g_{ik}\,, \tag{49}$$

where $\lambda_j$ is a vector.

If we impose simultaneously H. 4 and H. 5 both the equations (24) and (49) stand valid, so we get:

$$\lambda_i g_{jk} + \lambda_j g_{ki} + \lambda_k g_{ij} = 0, \tag{50}$$

contracting with $g^{jk}$ we get:

$$\lambda_i = 0, \tag{51}$$

i.e.

$$T_{ijk} = 0. \tag{52}$$

From now on we shall consider that (52) holds.

## 6. FINAL REMARKS ON THE TORSION.

If we were only interested in gravitational phenomena our work would also be finished. In fact, the torsion does not appear in the law of motion of free uncharged particles, so we can either choose it arbitrarily or consider it zero. The problem of the determination of the torsion can also be solved considering electromagnetic phenomena, without any attempt at a Unified Theory.

The electromagnetic laws, in Special Relativity, are:

$$\partial_i F^{ij} - J^j = 0 \quad , \quad \partial_i \overset{*}{F}{}^{ij} = 0. \tag{53}$$

We can find the corresponding laws in General Relativity, making use of H. 3, in the following way: let us try to calculate a vector $T_j$ given by:

$$T^j \equiv \nabla_i F^{ij} - J^j, \tag{54}$$

i.e.

$$T^j \equiv \partial_i F^{ij} + \left\{ {i \atop ih} \right\} F^{hj} + \left\{ {j \atop ih} \right\} F^{ih} + S_{ih}{}^{i} F^{hj} + S_{ih}{}^{j} F^{ih} - J^j. \tag{55}$$

In a local inertial frame, where $\left\{ {k \atop ij} \right\} = 0$, the equation (55) becomes (531) so:

$$T^i = S_{ih}{}^{i} F^{hj} + S_{ih}{}^{j} F^{ih}, \tag{56}$$

(5) In fact, all zeros of the polinomial (48) are also zeros of the four polinomials (47), $j = 1,2;\ 3,4$; being all the polinomials of second order. So we have that the coefficients of the four polinomials (47) are proportional to the coefficients of the polinomial (48), $\lambda_j$ are the constants of proportionality, and obviously they are the coordinates of a vector.

this is a tensorial equation therefore it is valid in all frames. Substituting this equation in (55) we have:

$$\partial_i F^{ij} + \left\{ {i \atop ih} \right\} F^{hj} + \left\{ {j \atop ih} \right\} F^{ih} - J^j = 0. \tag{57}$$

We see that the torsion does not appear because it cancels out in both sides of the equations. Thus the torsion can be ignored or more simply, can be considered null in the electromagnetic phenomena, whose laws in this case are:

$$\nabla_i F^{ij} - J^j = 0 \quad , \quad \nabla_i \overset{*}{F}{}^{ij} = 0. \tag{58}$$

## 7. Conclusion.

Based on H. 1, H. 2 and H. 3 we have demonstrated that the law of motion of a free particle is (15). H. 4 and H. 5 prove that the connection that appears in the derivative of (15) has $T_{ijk} = 0$. H. 4 and H. 5 can be based in the Strong Principle of Equivalence (cfr. [5], [6] and [7]). Besides the *Marzke-Wheller* Method of Measurement (cfr. [11]) suggests us an ideal experience to prove the constancy of $h$. The torsion is irrelevant for gravitational and electromagnetic phenomena, hence we can take $S_{ijk} = 0$. Thus the space-time has a structure of a Riemannian Manifold.

We have actually found a connection, i.e. a covariant derivative, so that the laws of our set of phenomena (e.g. (18), (30), (58)) can be obtained by the following *Rule of Transcription* (cfr. [4]).

*Every law of General Relativity* (valid in an arbitrary frame) *is obtained by the substitutions*

$$\partial_i \rightarrow \overset{\leftrightarrow}{\nabla}_i , \tag{59}$$

*from the corresponding law of Special Relativity* (valid in a local inertial frame and expressed in cartesian coordinates (6)).

(6) It is of interest to note that although the torsion is irrelevant, an arbitrary torsion does not satisfy the transcription rule, and produces non vanishing vectors in the right side of (58). It can be proved that the torsion, that satisfies this rule unambiguously, fulfils the following 24 equations:

$$F^{ij} S_{kij} = 0 \quad , \quad \overset{*}{F}{}^{ij} S_{kij} = 0,$$
$$F^{ij} S_{ijk} = 0 \quad , \quad \overset{*}{F}{}^{ij} S_{ijk} = 0,$$
$$F^{ij} S_{jk}{}^{k} = 0 \quad , \quad \overset{*}{F}{}^{ij} S_{jk}{}^{k} = 0.$$

The solution of this system of equations is:

$$S^{i,jk} = \sigma^{i\dot{\alpha}\lambda}\,\sigma^{j\dot{\beta}\mu}\,\sigma^{k\dot{\gamma}\nu}(\gamma_{\dot{\alpha}\dot{\beta}}\,S_{\dot{\gamma}\lambda\mu\nu} + \gamma_{\lambda\mu}\,S_{\nu\dot{\alpha}\dot{\beta}\dot{\gamma}}),$$

where $\sigma^{i\dot{\alpha}\lambda}$ are the *Pauli matrices*, $\gamma_{\lambda\mu}$ the fundamental antisymmetric spinor and $S_{\dot{\gamma}\lambda\mu\nu}$ is a symmetric spinor, in $\lambda, \mu, \nu$, that satisfies:

$$F^{\lambda\mu} S_{\dot{\gamma}\lambda\mu\nu} = 0,$$

where $F^{\lambda\mu}$ is the electromagnetic symmetric spinor that allows us to write the electromagnetic tensor as (cf. [1])

$$F^{ij} = \sigma^{i\dot{\alpha}\lambda}\,\sigma^{j\dot{\beta}\mu}(\gamma_{\lambda\mu}\,F_{\dot{\alpha}\dot{\beta}} + \gamma_{\dot{\alpha}\dot{\beta}}\,F_{\lambda\mu}).$$

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