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An example of generalized game derived from a finite game

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Teoria dei giochi. — *An example of generalized game derived from a finite game.* Nota di Ezio Marchi, presentata (*) dal Socio B. Segre.

Riassunto. — Si considera l'esistenza di punti di equilibrio di un esempio di giuoco generalizzato ottenuto dall'estensione mista di un giuoco finito fra un qualunque numero di persone con vincoli sulle strategie pure. Si presenta inoltre un secondo esempio interessante, dove qualsiasi composizione di punti di equilibrio delle parti è un punto corrispondente del giuoco generalizzato.

1. Let $\Gamma = \{\Sigma_1, \cdots, \Sigma_n; A_1, \cdots, A_n\}$ be a finite n-person game and for each player $i \in N$ let $\varphi_i : \Sigma_N = \underset{j \in N}{X} \Sigma_j \rightarrow \Sigma_i$ be a multivalued function whose image is a non-empty set for every $\sigma_N \in \Sigma_N$. Then, one can associate to it a generalized n-person game

$$\Lambda = \Lambda(\Gamma, \varphi) = \{\tilde{\Sigma}_1, \cdots, \tilde{\Sigma}_n; E_1, \cdots, E_n; Z_1, \cdots, Z_n\},$$

where the strategy sets are the mixed strategy sets, the payoffs are those expectations of the original game, and finally the restricted strategy set of the player $i \in N$ under the action of $x_N \in X_N = \underset{j \in N}{X} \tilde{\Sigma}_j$ is given by

$$Z_i(x_N) = \sum_{s_N} x_N(s_N) \widetilde{\varphi_i(s_N)} \subset \tilde{\Sigma}_i .$$

This set is convex since it is a linear combination of convex sets. When $\varphi_i(s_N) = \Sigma_i$ for all the joint pure strategies and all the players, Λ becomes the mixed extension $\tilde{\Gamma}$ of the original game.

If one sees the set $\varphi_i(\sigma_N)$ as the set of pure strategies that the player $i \in N$ is able to use under the action $\sigma_N \in \Sigma_N$ playing the game Γ, then the set $Z_i(x_N)$ is a natural representation for the set of permitted mixed behavior under the mixed action $x_N \in X_N$ in the statistical consideration.

Z_i *is upper-semicontinuous.* For any two given convergent sequences $x_N(k) \rightarrow x_N$ and $y_N(k) \in Z_i(x_N(k))$, there is for each k a $w_{s_N}(k) \in \widetilde{\varphi_i(s_N)}$ such that:

$$y_N(k) = \Sigma x_N(k)(s_N) w_{s_N}(k).$$

Let $w_{s_N}(k') \rightarrow \bar{w}_{s_N} \in \widetilde{\varphi_i(s_N)}$ be a convergent subsequence of $w_{s_N}(k)$, then

$$y_N(k) \rightarrow \Sigma x_N(s_N) \bar{w}_{s_N} \in Z_i(x_N).$$

Z_i *is lower-semicontinuous.* Let $x_N(k) \rightarrow x_N$ be a convergent sequence and a point $y_N = \Sigma x_N(s_N) w_{s_N} \in Z_i(x_N)$. Then the points $y_N(k) = \Sigma x_N(k)(s_N) w_{s_N}$ belong to the respective sets $Z_i(x_N(k))$.

(*) Nella seduta del 20 aprile 1968.

As a consequence of these properties and the multilinearity of the expectations functions, the function

$$\min_{w_{e(i)} \in Z_{e(i)}(x_N)} E_i(y_i, w_{e(i)}, x_{f(i)})$$

is continuous in the variable (y_i, x_N) belonging to the graph of Z_i and concave in $y_i \in Z_i(x_N)$ for any given sets

$$e(i) \subset N - \{i\} \qquad \text{and} \quad f(i) = N - (e(i) \cup \{i\}),$$

where $Z_{e(i)} = \underset{j \in e(i)}{X} Z_j$. Therefore, the set

$$S_i(x_N) = \{y_i \in Z_i(x_N) : \min_{w_{e(i)} \in Z_{e(i)}(x_N)} E_i(y_i, w_{e(i)}, x_{f(i)}) =$$

$$= \max_{w_i \in Z_i(x_N)} \min_{w_{e(i)} \in Z_{e(i)}(x)} E_i(w_i, w_{e(i)}, x_{f(i)})\}$$

is convex. This assures the following result.

THEOREM. *The e–generalized game* (Λ, e) *has an* e_m*–simple point, that is, a point* $\bar{x}_N \in X_N$ *such that* $\bar{x}_i \in S_i(\bar{x}_N)$ *for all* $i \in N$.

As an immediate consequence, when $e(i) = \varnothing$ for each player $i \in N$ we obtain the following:

COROLLARY. *The generalized game* Λ *has an equilibrium point, that is, a point* $x_N \in X_N$ *such that*

$$E_i(\bar{x}_N) = \max_{w_i \in Z_i(\bar{x}_N)} E_i(w_i, \bar{x}_{N-\{i\}})$$

and $\bar{x}_i \in Z_i(\bar{x}_N)$ *for all* $i \in N$.

In a similar way one could analyse the existence for other concepts as those introduced in [2].

2. We present now an interesting example of two-person game where some suitable *compositions* of equilibrium points of subgames are equilibrium points in the original game.

Let us consider the game Γ with $N = \{1, 2\}$, where each strategy set Σ_1 and Σ_2 are partitionated in $\Sigma_i^1, \cdots, \Sigma_i^r (i \in \{1, 2\})$ and the payoff function A_i is zero for $\sigma_1 \in \Sigma_1^{r_1}$ and $\sigma_2 \in \Sigma_2^{r_2}$ when $r_1 \neq r_2$. If the constrained actions of player i are independent on his own behavior and $\varphi_j(\sigma_i) = \Sigma_j^{r_i} (i \neq j)$ for $\sigma_i \in \Sigma_i^{r_i}$, then

$$Z_i(x_j) = \sum_{s_j \in \Sigma_j} x_j(s_j) \widetilde{\varphi_i(s_j)} = \sum_{k=1}^{r} s_k(x_j) \tilde{\Sigma}_i^k,$$

where $s_k(x_j) = \sum_{s_j \in \Sigma_j^k} x_j(s_j)$.

Let E_i^k be the expectation function of the restriction $A_i \mid \Sigma_i^k \times \Sigma_j^k$ defined on the set $\Sigma_i^k \times \Sigma_j^k$ considered as the product of two faces of X_N; then each mixed extension game

$$\Gamma_k = \{\tilde{\Sigma}_i^k , \tilde{\Sigma}_j^k ; E_i^k , E_j^k\},$$

for $k \in \{1 , \cdots , r\}$, has at least an equilibrium point $(\bar{x}_i^k , \bar{x}_j^k)$, that is, for $j \neq i \in \{1 , 2\}$,

$$E_i (\bar{x}_i^k , \bar{x}_j^k) \geq E_i (x_i^k , \bar{x}_j^k) \qquad \text{for all} \quad x_i^k \in \tilde{\Sigma}_i^k .$$

For any given numbers $0 \leq s_k \leq 1$ whose sum is 1, one obtains

$$\sum_k s_k^2 E_i (\bar{x}_i^k , \bar{x}_j^k) \geq \sum_k s_k^2 E_i (x_i^k , \bar{x}_j^k) \qquad \text{for all} \quad (x_i^1 , \cdots , x_i^r) ;$$

hence

$$E_i\Big(\sum_k s_k \bar{x}_i^k , \sum_k s_k \bar{x}_j^k\Big) \geq E_i\Big(\sum_k s_k x_i^k , \sum_k s_k \bar{x}_j^k\Big) \qquad \text{for all} \quad (x_i^1 , \cdots , x_i^r) ,$$

since the point $x_i = \sum_k s_k x_i^k$ is an element of Σ_i. In other words, this relation expresses that the point $(\bar{x}_i , \bar{x}_j)$ obtained by composition of those in the subgames is an equilibrium point of the generalized game Λ, as $s_k (\bar{x}_i) = s_k (\bar{x}_j) = s_k$ and therefore $\bar{x}_i \in Z_j (\bar{x}_j)$.

Furthermore, as an immediate consequence of this fact, it follows that, if $P_k \subset \tilde{\Sigma}_i^k \times \tilde{\Sigma}_j^k$ indicates the non-empty set of equilibrium points of game Γ_k, then the set $P = \underbrace{\sum s_k P_k}_{s \in \{1,\cdots,r\}} \subset X_{\{1,2\}}$ is a subset of equilibrium points of the generalized game Λ. So P_k is convex and then P is also convex when Γ is a zero sum game.

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