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Matematica. — *Sphere bundles over spheres which are* H–*spaces.*
Nota di ROY R. DOUGLAS e FRANÇOIS SIGRIST, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si completa un recente risultato di J. F. Adams, mostrando che nessun H–spazio può essere spazio totale in una fibrazione avente per fibra una 7–sfera e base una 11–sfera oppure una 15–sfera.

1. INTRODUCTION.

The Lie groups SO(3) and SU(3) provide examples of H–spaces which are sphere bundles over spheres. More generally, if an H–space G is a q–sphere bundle over the n–sphere (q and n positive), it follows from a theorem of J. F. Adams [1] that either both q and n belong to the set $\{1, 3, 7\}$, or (q, n) is one of the following four pairs:

$$(1, 2) \quad , \quad (3, 5) \quad , \quad (7, 11) \quad , \quad (7, 15).$$

Examples of such H–spaces G are known for each of the above pairs (q, n), with the exception of the pairs $(7, 11)$ and $(7, 15)$. It follows from the theorem below that *the remaining two pairs of dimensions*, $(7, 11)$ *and* $(7, 15)$, *are not possible.*

2. MAIN RESULT.

Suppose the integral cohomology ring $H^*(G)$ is a torsion-free, exterior algebra on two generators, and the dimensions of the generators are q and n, respectively.

THEOREM. *If* $(q, n) = (7, 11)$ *or* $(7, 15)$, *then* G *can not be an* H–*space.*

The two cases $(7, 11)$ and $(7, 15)$ require separate, though similar, proofs. However, we will present only a brief summary of the proof in case $(7, 11)$ in this note, because the argument in case $(7, 15)$ is significantly simpler than in case $(7, 11)$, and details of both proofs will appear elsewhere (1).

3. OUTLINE OF PROOF.

Let G be an H–space with torsion-free integral cohomology ring, $H^*(G)$, an exterior algebra on two generators, in dimensions seven and eleven, respectively. We will obtain a contradiction.

(*) Nella seduta del 20 aprile 1968.
(1) J. R. HUBBUCK, informs us that he has independently obtained this result.

Let X be the projective plane of (some H–space multiplication on) G. $H^*(X)$ is torsion-free. Let x and y be generators of $H^8(X)$ and $H^{12}(X)$, respectively. Then x and y generate a truncated polynomial algebra, truncated at height 3 (i.e., $x^3 = x^2 y = xy^2 = y^3 = 0$). (See [3]).

If $w \in H^*(X)$, we will also denote the mod 2 reduction of w by $w \in H^*(X; Z_2)$. The Steenrod algebra (mod 2) operates on the reduced truncated polynomial algebra in the following way.

$$\begin{aligned} Sq(x) &= x + y + x^2 \\ Sq(y) &= y + xy + y^2 \\ Sq(x^2) &= x^2 + y^2 \\ Sq(xy) &= xy + y^2 \\ Sq(y^2) &= y^2. \end{aligned}$$

By the usual spectral sequence argument, we know we can choose elements $\alpha, \beta \in \tilde{K}(X)$ such that their images under the Chern character

$$ch : \tilde{K}(X) \to \tilde{H}^*(X; Q)$$

(a ring monomorphism) are of the form (in dimensions $\leqq 24$):

$$\begin{aligned} ch(\alpha) &= x + \lambda_1 y + \lambda_2 x^2 + \lambda_3 xy + \lambda_4 y^2 \\ ch(\beta) &= y + \mu_1 x^2 + \mu_2 xy + \mu_3 y^2. \end{aligned}$$

Let our choice of $\alpha, \beta \in \tilde{K}(X)$ define the rational number $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mu_1, \mu_2, \mu_3$.

Let $m(2) = 2^2 \cdot 3$, $m(4) = 2^4 \cdot 3^2 \cdot 5$, $m(6) = 2^6 \cdot 3^3 \cdot 5 \cdot 7$ and $m(8) = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$. Then, by Theorem 3.5 of [4], it follows that

$$\begin{aligned} L_1 &= m(2) \cdot \lambda_1 \quad , \quad L_2 = m(4) \cdot \lambda_2 , \\ L_3 &= m(6) \cdot \lambda_3 \quad , \quad L_4 = m(8) \cdot \lambda_4 , \end{aligned}$$

and $M_2 = m(4) \cdot \mu_2$ are odd integers;

while $M_1 = \frac{1}{2} \cdot m(2) \cdot \mu_1$ and $M_3 = \frac{1}{2} \cdot m(6) \cdot \mu_3$ are integers.

If $\theta \in \tilde{K}(X)$, $ch_q(\theta)$ is the $2q$–dimensional component of $ch(\theta)$, and $\psi^k : K(X) \to K(X)$ are the Adams operation introduced in [2] (for each integer k), then

(A) $$ch_q(\psi^k(\theta)) = k^q \cdot ch_q(\theta)$$

(Theorem 5.1, [2]) and

(B) $$\psi^p(\theta) - \theta^p \equiv 0 \quad (\text{mod } p),$$

if p is a prime (Proposition 3.2.2, [5]).

Computing with (A), we obtain the following three lemmas. ($p = 2$ or 3).

LEMMA 1. *The coefficient of* β^2 *in* $\psi^p(\alpha)$ *is an integral multiple of* p *and equals*

$$p^4[(p^8-1)\lambda_4-(p^2-1)\lambda_1\mu_3+(p^4-1)\lambda_1^2\lambda_2-(p^2-1)\lambda_1^3\mu_1 -(p^6-1)\lambda_1\lambda_3+(p^2-1)\lambda_1^2\mu_2].$$

LEMMA 2. *The coefficient of* $\alpha\beta$ *in* $\psi^p(\alpha)$ *is an integral multiple of* p *and equals*

$$p^4[(p^6-1)\lambda_3-(p^2-1)\lambda_1\mu_2-2(p^4-1)\lambda_1\lambda_2+2(p^2-1)\lambda_1^2\mu_1].$$

LEMMA 3. *The coefficient of* β^2 *in* $\psi^3(\beta)$ *is an integral multiple of* 3 *and equals*

$$3^6[(3^6-1)\mu_3-(3^4-1)\mu_2\lambda_1+(3^2-1)\mu_1\lambda_1^2].$$

There exist integers x, y, z, t, u, v, such that

$$L_4 = 2x+1, \quad L_1L_3 = 2y+1,$$
$$L_1^2L_2 = 2z+1, \quad L_1^2M_2 = 2t+1,$$
$$L_1^3M_1 = u \quad \text{and} \quad L_1M_3 = v.$$

(N.B. Do not confuse integers x, y with cohomology classes x, y).

PROPOSITION. *The integers* x, y, z, t, u, v *satisfy*:

(C) $$4x+5y+6z+7t-3u-v \equiv 5 \pmod{16}$$

(D) $$x-5y+3z+7t-3u-v \equiv -3 \pmod{16}$$

(E) $$-y+4z-3t+2u \equiv 0 \pmod{4}$$

(F) $$3y-2z-t+2u \equiv 0 \pmod{4}$$

(G) $$-2t+u+v \equiv 1 \pmod{4}.$$

Proof of Proposition.

(C). Let $p = 3$ in Lemma 1. and multiply coefficient of β^2 by $2^5\cdot 5\cdot 7$ to get a congruence mod 32. This can be rewritten to give (C).

(D). Let $p = 2$ in Lemma 1. and multiply coefficient of β^2 by $2^4\cdot 3^3\cdot 5\cdot 7$ to get a congruence mod 32. This can be rewritten as (D).

(E). Let $p = 3$ in Lemma 2. and multiply the coefficient of $\alpha\beta$ by $2^3\cdot 5\cdot L_1$ to get a congruence mod 8, which is the same as (E).

(F). Let $p = 2$ in Lemma 2. and multiply the coefficient of $\alpha\beta$ by $2^2\cdot 3^2\cdot 5\cdot L_1$ to obtain a congruence mod 8, which is equivalent to (F).

(G). Multiply the coefficient of β^2 in $\psi^3(\beta)$ (see Lemma 3.) by $2^2\cdot 5\cdot L_1$. This gives the congruence mod 4 which we refer to as (G). q.e.d.

A computational argument (involving determinants) shows that the five congruences in the six variables do not have a set of simultaneous solutions (x, y, z, t, u, v) in the integers. This is a contradiction (2), and the theorem's proof is complete.

Remark. An interesting consequence of the above result concerns Stiefel manifolds. The manifold $W_{3,2}$ of (3×2) orthogonal quaternionic matrices, and the manifold $Y_{2,2}$ of square (2×2) orthogonal matrices over the Cayley numbers are q-sphere bundles over the n-sphere with $(q, n) = (7, 11)$ and $(7, 15)$, respectively. These two Stiefel manifolds are therefore not H-spaces.

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[2] J. F. ADAMS, *Vector fields on spheres,* « Ann. Math. », *75*, 603-632 (1962).
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[4] R. R. DOUGLAS, *Homotopy-commutativity in H-spaces,* « Quart. J. of Math. », *18*, 263-283 (1967).
[5] M. ATIYAH, *K-Theory* (W. A. Benjamin 1967).

(2) We obtain a contradiction at once by adding the three congruences (C), (F) and (G).