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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi matematica. — *On the non-linear wave equation with dissipative term discontinuous with respect to the velocity* (*). Nota I di LUIGI AMERIO e GIOVANNI PROUSE, presentata (**) dal Corrisp. L. AMERIO.

RIASSUNTO. — Si enuncia, in una conveniente classe funzionale, un teorema di unicità ed esistenza corrispondente al moto di una membrana m–dimensionale vibrante, sotto l'azione di una forza di massa e di una resistenza del mezzo funzione crescente (anche discontinua) della velocità. Si dà la dimostrazione del teorema di unicità.

1. – In the present paper we shall consider the wave equation with non-linear dissipative term

$$\text{(1.1)}\quad \sum_{j,k}^{1\cdots m} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\,\frac{\partial y(t,x)}{\partial x_k}\right) - a_0(x)\,y(t,x) - \frac{\partial^2 y(t,x)}{\partial t^2} + f(t,x) = \beta\left(\frac{\partial y(t,x)}{\partial t}\right)$$

being $x \in \Omega$ (open, bounded and connected set of R^m), $0 \le t \le T$. All functions considered are supposed to be real.

Equation (1.1) is a particular case of an abstract equation studied by Lions and Strauss (1). The general theorem proved by Lions and Strauss,

(*) Istituto Matematico del Politecnico di Milano. Gruppo di Ricerca n. 12 del Comitato per la Matematica del C.N.R.

(**) Nella seduta del 9 marzo 1968.

(1) J. L. LIONS and W. A. STRAUSS, *Some non–linear evolution equations*, « Bull. Soc. Math. France », 93 (1965).

applied to (1.1), guarantees the existence and uniqueness of a solution (in the large) of the initial-boundary value problem provided the function $\beta(\eta)$ is a non-decreasing function of class $C^1(R^1)$ with the same asymptotic behaviour as $\eta\,|\eta|^\sigma$, $\sigma \geq 0$ (2).

Our aim is to show that such an existence-uniqueness theorem continues to hold assuming only that $\beta(\eta)$ is non-decreasing (and is not, therefore, necessarily continuous).

We shall assume that the coefficients $a_{jk}(x)$, $a_0(x)$ are measurable and bounded on Ω and satisfy the conditions

$$\nu \sum_{1}^{m}{}_j\, \xi_j^2 \leq \sum_{j,k}^{1\cdots m} a_{jk}(x)\,\xi_j\,\xi_k \qquad \forall\,(\xi_1,\cdots,\xi_m) \in R^m \; (\nu > 0), \tag{1.2}$$

$$a_{jk}(x) = a_{kj}(x) \quad , \quad a_0(x) \geq 0. \tag{1.3}$$

We shall, moreover, assume that

$$\int_0^T \left\{ \int_\Omega f^2(t,x)\,dx \right\}^{1/2} dt < +\infty. \tag{1.4}$$

Let $\beta(\eta)$ be a function of the variable η, *non-decreasing on an interval* $a \overline{\quad} b$, *with* $-\infty \leq a < 0 < b \leq +\infty$. We assume moreover that

$$\lim_{\eta\to b^-} \beta(\eta) = +\infty \qquad \text{if} \quad b < +\infty,$$

$$\lim_{\eta\to a^+} \beta(\eta) = -\infty \qquad \text{if} \quad a > -\infty.$$

The function $\beta(\eta)$ *can therefore have discontinuities of the first kind* in a sequence of points $\{\eta_s\}$; for the time being we shall not define the values $\beta(\eta_s)$; moreover we shall assume that $\beta(0^-) \leq 0$, $\beta(0^+) \geq 0$.

It is, in fact, necessary to state precisely what is meant by "solution" of (1.1) in the cylinder $Q = 0 \vert\!\overline{\quad}\!\vert T \times \Omega$.

We shall say that $y(t,x)$ is a *solution* of (1.1) in Q if the following conditions are satisfied:

1) $y(t,x) \in H^1(Q)$;

2) *the distributions* $A(x)\,y(t,x) = \sum_{j,k}^{1\cdots m} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\,\frac{\partial y(t,x)}{\partial x_k}\right) - a_0(x)\,y(t,x)$, $\frac{\partial^2 y(t,x)}{\partial t^2}$ *are functions* $\in L^2(Q)$;

3) *it results, almost everywhere on* Q,

$$a < y_t(t,x) < b, \tag{1.5}$$

$$A(x)\,y(t,x) - y_{tt}(t,x) + f(t,x) \in \beta(y_t(t,x)^-) \vert\!\overline{\quad}\!\vert\, \beta(y_t(t,x)^+). \tag{1.6}$$

(2) This hypothesis has recently been substituted, in the case $m=1$, by Buzzetti with the one that $\beta(\eta)$ be non-decreasing and of class $C^1(R^1)$ (F. BUZZETTI, *Sull'equazione della corda vibrante con termine dissipativo monotono crescente*, «Rend. Ist. Lomb. Sc. Lett.» (1968)).

Relation (1.6) means that, if at the point $(\bar{t}, \bar{x})$ it is $\frac{\partial y(\bar{t}, \bar{x})}{\partial t} = \bar{\eta}$ and if $\beta(\eta)$ is continuous in $\bar{\eta}$, then in $(\bar{t}, \bar{x})$ equation (1.1) holds; if instead $\bar{\eta} = \eta_s$, then it must be

$$\beta(\eta_s^-) \leq \sum_{j,k}^{1\cdots m} \frac{\partial}{\partial x_j}\left(a_{jk}(\bar{x}) \frac{\partial y(\bar{t}, \bar{x})}{\partial x_k}\right) - a_0(\bar{x})\, y(\bar{t}, \bar{x}) -$$

$$- \frac{\partial^2 y(\bar{t}, \bar{x})}{\partial t^2} + f(\bar{t}, \bar{x}) \leq \beta(\eta_s^+).$$

Let us observe that (1.6) may be written in the form (1.1) provided it is understood that *at the point* η_s *the function* $\beta(\eta)$ *may take on any value of the interval* $\beta(\eta_s^-) \vdash\!\dashv \beta(\eta_s^+)$. We shall therefore, by this convention, always consider the equation in the form (1.1).

We shall consider, for (1.1), the classical mixed problem, which consists in finding a solution $y(t, x)$, defined on Q, satisfying the boundary condition

$$(1.7) \qquad y(t, x)\Big|_{x \in \partial\Omega} = 0 \qquad (0 \leq t \leq T)$$

and the initial conditions

$$(1.8) \qquad y(0, x) = u_0(x) \quad , \quad y_t(0, x) = u_1(x) \qquad (x \in \Omega).$$

In other words, we shall study the motion of a vibrating membrane with fixed boundary, under the action of an external force $f(t, x)$ and of a resistance, which is an increasing (and eventually also discontinuous) function of the velocity; moreover, if b is finite, the velocity cannot exceed the value b (corresponding to an infinite resistance), and analogously for a. The problem here treated is obviously of interest to the *theory of controls*.

Setting

$$y(t) = \{y(t, x); x \in \Omega\} \quad , \quad y'(t) = \left\{\frac{\partial y(t, x)}{\partial t}; x \in \Omega\right\},$$

$$y''(t) = \left\{\frac{\partial^2 y(t, x)}{\partial t^2}; x \in \Omega\right\},$$

$$\mathrm{A}y(t) = \{\mathrm{A}(x)\, y(t, x); x \in \Omega\},$$

$$f(t) = \{f(t, x); x \in \Omega\} \quad , \quad \beta(y'(t)) = \left\{\beta\left(\frac{\partial y(t, x)}{\partial t}\right); x \in \Omega\right\},$$

equation (1.1) may be written in operational form

$$(1.9) \qquad \mathrm{A}y(t) - y''(t) + f(t) = \beta(y'(t)).$$

In what follows we shall utilize systematically the classical Hilbert spaces $L^2 = L^2(\Omega)$, $H_0^1 = H_0^1(\Omega)$, $E = H_0^1 \times L^2$ (energy space). If $u(t)$, $u'(t)$ take their values respectively in H_0^1 and L^2, we shall set, as usual,

$$\|u(t)\|_E = \{\|u(t)\|_{H_0^1}^2 + \|u'(t)\|_{L^2}^2\}^{1/2}$$

where

$$\|u(t)\|_{H_0^1} = \left\{ \int_\Omega \left(\sum_{j,k}^{1\cdots m} a_{jk}(x) \frac{\partial u(t,x)}{\partial x_j} \frac{\partial u(t,x)}{\partial x_k} + a_0(x) u^2(t,x) \right) dx \right\}^{1/2}.$$

Let $\{g_j\}$ be the sequence of the eigensolutions of the operator A $\left(g_j \in H_0^1, (g_j, g_k)_{L^2} = \left(\frac{g_j}{\sqrt{\lambda_j}}, \frac{g_k}{\sqrt{\lambda_k}}\right)_{H_0^1} = \delta_{jk}\right)$ and $\{\lambda_j\}$ $\left(\lambda_{j+1} \geq \lambda_j > 0, \lim_{j\to\infty} \lambda_j = +\infty\right)$ the sequence of the corresponding eigenvalues. If $y \in L^2$, it is

$$(1.10) \qquad y = \sum_1^\infty {}_n (y, g_n)_{L^2} g_n \quad , \quad \sum_1^\infty {}_n (y, g_n)^2 = \|y\|_{L^2}^2 < +\infty.$$

Characteristic condition for y to belong to H_0^1 is that

$$(1.11) \qquad y = \sum_1^\infty {}_n \left(y, \frac{g_n}{\sqrt{\lambda_n}}\right)_{H_0^1} \frac{g_n}{\sqrt{\lambda_n}} = \sum_1^\infty {}_n \sqrt{\lambda_n} (y, g_n)_{L^2} \frac{g_n}{\sqrt{\lambda_n}},$$

$$\|y\|_{H_0^1}^2 = \sum_1^\infty {}_n \lambda_n (y, g_n)_{L^2}^2 < +\infty.$$

Observe now that, $\forall \varphi \in \mathfrak{D}(\Omega)$, it is $(Ay, \varphi)_{L^2} = \langle Ay, \varphi \rangle = -(y, \varphi)_{H_0^1}$; hence, $\forall g \in H_0^1$, $(Ay, g)_{L^2} = -(y, g)_{H_0^1}$.

Characteristic condition for $Ay \in L^2$ and $y \in H_0^1$ is that

$$(1.12) \qquad Ay = \sum_1^\infty {}_n (Ay, g_n)_{L^2} g_n = -\sum_1^\infty {}_n (y, g_n)_{H_0^1} g_n = -\sum_1^\infty {}_n \lambda_n (y, g_n)_{L^2} g_n$$

that is

$$(1.13) \qquad \|Ay\|_{L^2}^2 = \sum_1^\infty {}_n \lambda_n^2 (y, g_n)_{L^2}^2 < +\infty.$$

Hence, if, for a sequence $\{y_k\}$, it results $\|Ay_k\|_{L^2} \leq M$, the sequence itself is relatively compact in the norm of H_0^1.

Let us denote by Γ the class of functions $z(t) = \{z(t, x); x \in \Omega\}$ such that:

a) $z(t) \in C^0(0 \vdash\!\dashv T; E)$;

b) $z'(t) \in L^\infty(0 \vdash\!\dashv T; E)$;

c) $Az(t) \in L^\infty(0 \vdash\!\dashv T; L^2)$.

Hence $z(t) \in \Gamma$ means that $z(t), z'(t)$ are continuous from $0 \vdash\!\dashv T$ to H_0^1 and L^2 respectively, that $z'(t)$ and $z''(t)$ are bounded from $0 \vdash\!\dashv T$ to H_0^1 and L^2 respectively and that $Az(t)$ is bounded from $0 \vdash\!\dashv T$ to L^2.

The problem defined by (1.7), (1.8) corresponds to the following initial value problem for equation (1.9): *given u_0 and u_1 find a solution $y(t)$ satisfying the initial conditions*

$$(1.14) \qquad y(0) = u_0 \quad , \quad y'(0) = u_1.$$

Setting

$$(1.15)\qquad \bar\beta(\eta) = \begin{cases} 0 & \text{when } \eta = 0 \\ \beta(\eta^-) & \text{when } 0 < \eta < b \\ \beta(\eta^+) & \text{when } a < \eta < 0, \end{cases}$$

we shall prove, in the present paper, the following result.

Assume that:

1) $u_0 \in H_0^1$, $Au_0 \in L^2$;

2) $u_1 \in H_0^1$, $a < u_1(x) < b$ (almost–everywhere on Ω), $\bar\beta(u_1) \in L^2$;

3) $f(0) \in L^2$, $f'(t) \in L^1(0 \vdash\!\dashv T; L^2)$.

Then problem (1.9), (1.14) *admits, in the functional class* Γ, *one and only one solution.*

2. – Let us prove the uniqueness theorem.

Problem (1.9), (1.14) *has, in* Γ, *at most one solution.*

Let, in fact, $y(t)$ and $z(t)$ be two solutions $\in \Gamma$, such that

$$(2.1)\qquad y(0) = z(0) = u_0 \quad , \quad y'(0) = z'(0) = u_1 .$$

It is then

$$(2.2)\qquad \begin{aligned} Ay(t) - y''(t) + f(t) &= \beta(y'(t)) \\ Az(t) - z''(t) + f(t) &= \beta(z'(t)) \end{aligned}$$

and, setting $w(t) = z(t) - y(t)$,

$$(2.3)\qquad Aw(t) - w''(t) = \beta(y'(t) + w'(t)) - \beta(y'(t)),$$

where $w(t)$ satisfies the initial conditions

$$(2.4)\qquad w(0) = 0 \quad , \quad w'(0) = 0.$$

Let us consider the scalar product, in L^2, of (2.3) by $w'(t)$ and integrate between 0 and $t \in 0 \vdash\!\dashv T$. It results, bearing in mind (2.4) and being, by (2.2), $\beta(y'(t))$, $\beta(z'(t)) \in L^\infty(0 \vdash\!\dashv T; L^2)$,

$$\|w(t)\|_E^2 = -2 \int_0^t (\beta(y'(\eta) + w'(\eta)) - \beta(y'(\eta)), w'(\eta))_{L^2}\, d\eta =$$

$$= -2 \int_0^t d\eta \int_\Omega (\beta(y_t(\eta, x) + w_t(\eta, x)) - \beta(y_t(\eta, x)))\, w_t(\eta, x)\, dx.$$

Observe now that it results, almost everywhere on Q,

$$(2.6)\qquad (\beta(y_t(\eta, x) + w_t(\eta, x)) - \beta(y_t(\eta, x)))\, w_t(\eta, x) \geq 0.$$

This obviously occurs in a point $(\bar{\eta}, \bar{x})$ if it is $w_t(\bar{\eta}, \bar{x}) = 0$.

Assume now, for instance, $w_t(\bar{\eta}, \bar{x}) > 0$; setting then

$$\eta_1 = y_t(\bar{\eta}, \bar{x}) \quad , \quad \eta_2 = y_t(\bar{\eta}, \bar{x}) + w_t(\bar{\eta}, \bar{x}),$$

it is (being $\eta_1 < \eta_2$)

$$\beta(\eta_1) \leq \beta(\eta_1^+) \leq \beta(\eta_2^-) \leq \beta(\eta_2).$$

Hence, $\forall w_t(\eta, x)$, (2.6) is proved.

It follows, by (2.5),

$$\|w(t)\|_E = 0 \qquad \text{when} \quad 0 \leq t \leq T. \tag{2.7}$$

The theorem is therefore proved.

Let us now add an obvious inequality, which will be utilized in the proof of the existence theorems we shall give at § 2.

Let $0 \leq \varphi(t) \in L^\infty(0 \vdash\!\dashv T)$, $0 \leq \omega(t) \in L^1(0 \vdash\!\dashv T)$, *and, moreover,*

$$\varphi^2(t) \leq K^2 + 2\int_0^t \omega(\eta)\,\varphi(\eta)\,d\eta.$$

It results then

$$\varphi^2(t) \leq 4\left\{K^2 + \left(\int_0^t \omega(\eta)\,d\eta\right)^2\right\}. \tag{2.8}$$

Setting in fact $\mu(t) = \sup_{0 \leq \eta \leq t} \varphi(\eta)$, it is

$$\varphi^2(t) \leq K^2 + 2\,\mu(t)\int_0^t \omega(\eta)\,d\eta.$$

Hence

$$\varphi^2(\tau) \leq K^2 + 2\,\mu(t)\int_0^t \omega(\eta)\,d\eta, \qquad \forall \tau\; 0 \vdash\!\dashv t$$

and, consequently,

$$\mu^2(t) \leq K^2 + 2\,\mu(t)\int_0^t \omega(\eta)\,d\eta,$$

from which follows (2.8).