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The Lorentz Group over a finite field, and related properties of Dirac Spinors

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Fisica del discreto. — *The Lorentz Group over a finite field, and related properties of Dirac Spinors.* Nota di ENRICO G. BELTRAMETTI e ALBERTO BLASI, presentata (*) dal Socio B. SEGRE.

SUNTO. — Si esaminano le rappresentazioni del gruppo di Lorentz proprio ed improprio, si introducono gli spinori di Dirac e le forme sesquilineari associate con speciale riguardo alle correnti vettoriali e vettoriali–assiali: tutto ciò in relazione ad una geometria di Galois d'ordine p primo e $\equiv 3 \pmod 4$. Le dimostrazioni verranno date assieme ad ulteriori sviluppi in due lavori di prossima pubblicazione.

1. – INTRODUCTION.

In the present note we report, omitting details and explicit proofs, some new results in the description of physical schemes within a Galois geometry. For a treatment of old and new results on these geometries, as well as for an extensive bibliography on them, we refer to a recent paper by B. Segre [1]. The way to approximate (with arbitrary accuracy) the ordinary continuous geometry has been examined in the literature [2] and in the field of physics some attractive consequences of interpreting the space-time manifold as a Galois geometry have been pointed out [3, 4, 5]. Apart from the absence of divergences in field theories, there appear a number of symmetry properties due to the very fact of finiteness of space and independent of the number of points in it. We deal here with such symmetry properties, obtaining results which exhibit some connections with the papers of H. R. Coish [3] and I. S. Shapiro [4].

2. – ROTATION AND LORENTZ GROUPS IN A GALOIS GEOMETRY.

Let GF (p) be the primitive Galois field of prime order p: as usual the "zero" element under addition will be denoted by 0, and the integral marks of GF (p) will be denoted by the corresponding integer number symbol. The element -1, defined by $-1+(1)=0$, is a non-square element for suitable choices [2] of p, and precisely for $p \equiv 3 \pmod 4$, as we here assume. When the space-time coordinates take values in GF (p), one is led to define the finite version L $(4, p)$ of the proper Lorentz group as the group of invertible linear substitutions

$$x'_\mu = \sum_\nu l_{\mu\nu}\, x_\nu \quad ; \quad \mu, \nu = 0, 1, 2, 3 \quad ; \quad x_\mu, l_{\mu\nu} \in \mathrm{GF}(p) \quad ; \quad \det l = +1 \text{ (**)},$$

which leave $x_0^2 - x_1^2 - x_2^2 - x_3^2$ invariant; its order turns out to be [6] $\Omega_{L(4,p)} = p^2(p^4-1)$.

(*) Nella seduta del 9 marzo 1968.

(**) l stands for the 4×4 matrix whose elements are the $l_{\mu\nu}$ coefficients.

Similarly, the proper three-dimensional rotation subgroup $R(3, p)$ is formed by the substitutions

$$x'_i = \sum_j r_{ij} x_j \quad ; \quad i, j = 1, 2, 3 \quad ; \quad x_i, r_{ij} \in GF(p) \quad ; \quad \det r = +1,$$

leaving $x_1^2 + x_2^2 + x_3^2$ invariant.

To classify the representations of these groups it is useful to study the homomorphism with the linear group of 2×2 matrices with complex elements $(x + iy) \in GF(p^2)$, with $x, y \in GF(p)$, $i \in GF(p^2)$, $i^2 = -1$.

Let us introduce the group $SL^{(\pm)}(2, p^2)$ of 2×2 matrices over $GF(p^2)$

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL^{(\pm)}(2, p^2) \quad ; \quad \alpha, \beta, \gamma, \delta \in GF(p^2) \quad ; \quad \det a = \pm 1;$$

it is a finite group of order $2\,\Omega_{L(4,p)}$. Setting $\hat{x} = \sum_\mu \sigma_\mu x_\mu$, where σ_1, σ_2, σ_3 are the usual Pauli matrices and $\sigma_0 = \mathbf{1}$, one can verify that

$$\hat{x}' = (\det a)\, a\, \hat{x}\, a^+, \qquad (a^+ = \text{hermitian conjugate of } a) \tag{1}$$

induces a proper Lorentz transformation "l" on the coordinates x_μ. Clearly a and $-a$ correspond to the same $l \in L(4, p)$ and it may be shown that the correspondence may be inverted, thus determining a 1 to 2 homomorphism of $L(4, p)$ onto $SL^{(\pm)}(2, p^2)$. This homomorphism carries $R(3, p)$ onto the subgroup $SU^{(\pm)}(2, p^2)$ of matrices

$$u = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \in SU^{(\pm)}(2, p^2) \quad ; \quad \alpha, \beta \in GF(p^2) \quad ; \quad \det u = \pm 1,$$
$$u\, u^+ = (\det u)\, \mathbf{1}\,.$$

Due to the appearance of both signs in the determinant of the 2×2 matrices, a new two-valued label is needed, besides the usual quantum numbers, to specify the representations of the corresponding groups. In fact, all irreducible inequivalent representations of $SL^{(\pm)}(2, p^2)$ have the explicit form:

$$D^{(j,k;e)}(a) = (\det a)^e\, D^{(j)}(a) \otimes (D^{(k)}(a))^*, \quad \begin{cases} j, k = 0, 1/2, 1, \cdots \frac{p-1}{2} \\ e = 0, 1 \end{cases} \tag{2}$$

denoting by $D^{(j)}(a)$ a matrix of order $(2j + 1)$ with elements

$$D^{(j)}_{m',m}(a) = \frac{N^{(j)}_m}{N^{(j)}_{m'}} \sum_{k=\mathrm{Max}(0,m-m')}^{\mathrm{Min}(j+m,j-m')} \binom{j+m}{k} \binom{j-m}{k+m'-m} \cdot \tag{3}$$
$$\cdot\, \alpha^{j-m'-k}\, \delta^{j+m-k}\, \beta^k\, \gamma^{k-m+m'},$$

where $N^{(j)}_m \in GF(p^2)$ is a normalization factor to be specified later. We remark that the range of the integer or half integer labels j, k is bounded by the value $\frac{p-1}{2}$: a larger value of j or k would give rise to reducible representations; the number of representations given by Eqs. (2), (3) is thus $2p^2$ and, besides them, no other irreducible inequivalent representation can exist, since $2p^2$ is just the number of equivalent classes of $SL^{(\pm)}(2, p^2)$.

All irreducible inequivalent representation of $SU^{(\pm)}(2, p^2)$ may now be written as $D^{(j,0;e)}(u)$, putting $\delta = \alpha^*, \gamma = -\beta^*$ in Eq. (3): $D^{(j,0;e)}(u)$ is equivalent to $D^{(0,j;e)}(u)$.

As in the classical case, we require, on physical grounds, the irreducible representations of the Lorentz group to become unitary when restricted to the three-dimensional rotation group. One has thus to check the unitary character of $D^{(j,0;e)}(u)$. Choosing $N_m^{(j)}$ in (3) such that $N_m^{(j)} \cdot N_m^{(j)*} = \frac{(2j)!}{(j+m)!\,(j-m)!}$ it is easily found

$$(4) \qquad (D^{(j,0;e)}(u))^+ = (\det u)^{2j}\,(D^{(j,0;e)}(u))^{-1} = (\det u)^{2j}\,(D^{(j,0;e)}(u^{-1})),$$

so that $D^{(j,0;e)}(u)$ is unitary for integer j, but it is not for half-integer j.

At first sight one could guess that, owing to the finite order of the group, the non unitary representations could be made unitary by a suitable equivalence transformation; this is not the case since the classical theorem ensuring this possibility is no longer valid for representations built over a finite field (modular representations).

However, this problem has here an easy solution since unitarity may be recovered by letting the representation become a ray representation [7], (actually, to describe the transformation laws of physical states in quantum mechanics, one only needs ray representations). In fact, replacing $D^{(j,0;e)}(a)$ in Eq. (2) by

$$(5) \qquad \tilde{D}^{(j,0;e)}(a) = (\det a)^e \cdot D^{(j,0;e)}(\chi_a\, a),$$

where χ_a is a solution of

$$(6) \qquad \chi_a\, \chi_a^* = \det a \quad , \quad \chi_a \in GF(p^2),$$

one obtains a ray representation, with multiplication rule

$$\tilde{D}^{(j,0;e)}(a)\,\tilde{D}^{(j,0;e)}(a') = \omega_{a,a'}^{(j)}\, D^{(j,0;e)}(aa') \quad , \quad \omega_{a,a'}^{(j)} = \left(\frac{\chi_a\,\chi_{a'}}{\chi_{aa'}}\right)^{2j}.$$

By allowing χ_a to assume different solutions of (6), one obtains ray representations of the same equivalence class; the same happens going from $e = 0$ to $e = 1$: one may thus assume, as a representative of the equivalence class,

$$\tilde{D}^{(j,0)}(a) = D^{(j)}(\chi_a\, a)$$

where the label e has been omitted, χ_a is a fixed solution of (6) and $D^{(j)}$ is given by Eq. (3). The ray representations so defined exhibit the correct unitarity properties, when restricted to $SU^{(\pm)}(2, p^2)$ (see Eq. (4)).

3. – SPINOR REPRESENTATIONS OF THE EXTENDED LORENTZ GROUP $\mathfrak{L}(4, p)$.

We now want to adjoin the space reflection operation P to the proper group L $(4, p)$, thus obtaining the extended Lorentz group $\mathfrak{L}(4, p)$. Acting with P on the coordinates, $\hat{x}$ is transformed non linearly into

$$\hat{x}^{(\mathrm{P})} = \varepsilon (\hat{x})^{*} \varepsilon^{-1} \quad , \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

To have a (linear) representation of P we need to adopt the 4×4 basis

$$\hat{\mathrm{X}} = \begin{pmatrix} \hat{x} & 0 \\ 0 & \hat{x}^{(\mathrm{P})} \end{pmatrix}, \tag{7}$$

on which P induces the transformation

$$\hat{\mathrm{X}} \rightarrow \hat{\mathrm{X}}' = \begin{pmatrix} \hat{x}^{(\mathrm{P})} & 0 \\ 0 & \hat{x} \end{pmatrix} = \gamma_0 \hat{\mathrm{X}} \gamma_0$$

with

$$\gamma_0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

We have now to examine how the proper group L $(4, p)$ is represented on the new basis (7).

Hereinafter we shall consider only the case $j = 1/2$, and we remark that Eq. (1) may be rewritten as $\hat{x}' = \tilde{\mathrm{D}}^{(1/2,0)}(a)\, \hat{x}\, (\tilde{\mathrm{D}}^{(1/2,0)}(a))^{+}$.

The transformation induced on $\hat{\mathrm{X}}$ will then have the form

$$\hat{\mathrm{X}} \rightarrow \hat{\mathrm{X}}' = \mathrm{S}(a)\, \hat{\mathrm{X}} \mathrm{S}(a)^{+}$$

with $\mathrm{S}(a) = \begin{pmatrix} \tilde{\mathrm{D}}^{(1/2,0)}(a) & 0 \\ 0 & \theta_a \varepsilon \tilde{\mathrm{D}}^{(0,1/2)}(a) \varepsilon^{-1} \end{pmatrix}$.

Due to the fact that S (a) is required to be a ray representation as a whole, the relative phase θ_a can assume two values

$$\theta_a^{(e)} = (\det a)^e, \qquad e = 0, 1,$$

which give inequivalent ray representations.

Hence the spinor representations of the extended group $\mathfrak{L}(4, p)$ can be written as $(\gamma_0, \mathrm{S}^{(e)}(a))$ with $e = 0, 1$, depending on the choice of the phase θ_a; accordingly two kinds of "Dirac fields" $\psi^{(0)}$ and $\psi^{(1)}$ are possible.

4. – COVARIANT CURRENTS.

Consider the bi-spinor sesquilinear currents

$$\psi^{(e)+} \mathrm{B}_{(\tau)} \varphi^{(e)} \tag{8}$$

where $\psi^{(e)}, \varphi^{(e)}$ are Dirac spinors which transform according to the representation $(\gamma_0, \mathrm{S}^{(e)}(a))$, and $\mathrm{B}_{(\tau)}$ is a 4×4 matrix whose tensor nature is specified by a set of indices shortly denoted by (τ). We shall now examine whether $B_{(\tau)}$ matrices exist which make the current (8) covariant with respect to $\mathfrak{L}(4, p)$. As a guide, let us recall that in the classical continuous case the analogous question leads to the construction of five covariant currents, i.e., a scalar, a pseudo-scalar, a vector and an axial-vector, a second order tensor.

The general transformation law for the current (8) is:

$$\psi^{(e)^+} B_{(\tau)} \varphi^{(e)} \to \psi^{(e)^+} \gamma_0 B_{(\tau)} \gamma_0 \varphi^{(e)} = \psi^{(e)^+} B^{(P)}_{(\tau)} \varphi^{(e)}, \quad \text{under P, and}$$

$$\psi^{(e)^+} B_{(\tau)} \varphi^{(e)} \to \psi^{(e)^+} S^{(e)^+}(a) B_{(\tau)} S^{(e)}(a) \varphi^{(e)} = \psi^{(e)^+} B^{(L)}_{(\tau)} \varphi^{(e)} \quad \text{under } L(4, p).$$

The covariance requirements for (8) give relations between $B^{(P)}_{(\tau)}$, $B^{(L)}_{(\tau)}$, $B_{(\tau)}$ in which the label e appears explicitly; this accounts for differences between the present and the classical case. Actually we get:

(i) *Scalar*—In this case $B_{(\tau)}$ is a scalar matrix B, and the current (8) is required to be invariant both for P and $L(4, p)$: hence $B^{(P)} = B = B^{(L)}$. The equations must be verified identically with respect to a, and the only solution is given by

$$e = 0 \quad ; \quad B = k_S \gamma_0 \quad , \quad k_S \in GF(p^2).$$

(ii) *Pseudo-scalar*—The current (8) is now required to be invariant for $L(4, p)$ while it should change sign under the operation P, i.e., $-B^{(P)} = B = B^{(L)}$.

As a unique solution we find

$$e = 0 \quad ; \quad B = k_{PS} \gamma_0 \gamma_5 \quad , \quad k_{PS} \in GF(p^2),$$

where we have set

$$\gamma_5 = i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(iii) *Vector*—In this case $B_{(\tau)}$ is a four-vector, (τ) labels its components and will be replaced by $\mu = 0, 1, 2, 3$. The current (8) is required to transform like the coordinates x_μ both for P and $L(4, p)$, i.e.

$$B^{(P)}_\mu = -(-1)^{\delta_{0\mu}} \cdot B_\mu,$$

$$B^{(L)}_\mu = \frac{1}{2} (\det a) \sum_{\nu=0}^{3} Sp(\sigma_\mu a \sigma_\nu a^+) B_\nu.$$

For both $e = 0$ and $e = 1$ we obtain the solution

$$e = 0, 1 \quad ; \quad B_\mu = k_V \gamma_0 \gamma_\mu \quad , \quad k_V \in GF(p^2),$$

where we have set

$$\gamma_l = \begin{pmatrix} 0 & -\sigma_l \\ \sigma_l & 0 \end{pmatrix}, \qquad l = 1, 2, 3.$$

(iv) *Axial-vector*—The current (8) is now required to transform like the coordinates for $L(4, p)$ while for the operation P we get $B_\mu^{(P)} = (-1)^{\delta_{0\mu}} B_\mu$. For both $e = 0$ and $e = 1$ we arrive at the solution $e = 0, 1$;

$$B_\mu = k_A \gamma_0 \gamma_5 \gamma_\mu \quad , \quad k_A \in GF(p^2).$$

(v) *Tensor*—In this case $B_{(\tau)}$ is a 2nd-order tensor, (τ) will be replaced by two indices $\mu, \nu = 0, 1, 2, 3$. The current (8) then transforms under P according to

$$B_{\mu\nu}^{(P)} = (-1)^{\delta_{0\mu}+\delta_{0\nu}} B_{\mu\nu},$$

while for $L(4, p)$ the transformation law is

$$B_{\mu\nu}^{(L)} = \frac{1}{4} \sum_{\varrho,\lambda=0}^{3} Sp(\sigma_\mu a \sigma_\varrho a^+) Sp(\sigma_\nu a \sigma_\lambda a^+) B_{\varrho\lambda}.$$

In this case the solution is:

$$e = 0 \quad ; \quad B_{\mu\nu} = k_T \gamma_0 \gamma_\mu \gamma_\nu \quad , \quad k_T \in GF(p^2).$$

The γ matrices we have defined above obey the relations

$$\left.\begin{aligned} &\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} \\ &\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \end{aligned}\right\} \quad , \quad \mu, \nu = 0, 1, 2, 3, \quad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

thus they provide a realization of Dirac matrices which coincides, up to a factor $(-i)$, with Weyl's representation.

Summarizing, spinors of the kind $e = 0$ allow the construction of all usual covariant currents, while, with spinors of the kind $e = 1$, only vector (V) and axial-vector (A) currents are possible.

Currents of the type $\psi^{(e)+} B_{(\tau)} \varphi^{(e')}$ with $e \neq e'$ need not be taken into account for it can be easily proved that matrices $B_{(\tau)}$, making them covariant do not exist.

Referring to the construction of hamiltonians which occur in weak interaction theory, the problem arises of multiplying two currents to obtain a scalar or a pseudo-scalar quantity.

If one tentatively associates the leptons to the choice $e = 1$, the V, A currents appearing in weak interactions would be the unique possibility in the framework of a finite geometry.

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[6] See e.g. L. DICKSON, *Linear Groups* (Dover Publ. Inc. New York 1958).

[7] See e.g. M. HAMERMESH, *Group Theory* (Addison–Wesley 1962 –, Ch. 12).