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Geometria differenziale. — *Various projective invariants in a conformal Finsler space.* Nota di H. D. PANDE, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Si esaminano gli invarianti proiettivi di due spazi di Finsler in corrispondenza conforme.

1. CONFORMAL FINSLER SPACES.

Let two distinct metric functions $F(x, \dot{x})$ and $\bar{F}(x, \dot{x})$ be defined over an n-dimensional space F_n which satisfy the requisite conditions for a Finsler space[1] (1). The two metrics resulting from them are called conformal if the corresponding metric tensors $g_{ij}(x, \dot{x})$ and $\bar{g}_{ij}(x, \dot{x})$ are proportional to each other. Knebelman[2] has shown that the factor of proportionality between them is at most a point function. Thus, we have [1].

(1.1) $\bar{g}_{ij}(x, \dot{x}) = e^{2\sigma} g_{ij}(x, \dot{x})$, (1.2) $\bar{g}^{ij}(x, \dot{x}) = \bar{e}^{2\sigma} g^{ij}(x, \dot{x})$,

and

$$\bar{F}(x, \dot{x}) = e^{\sigma} F(x, \dot{x}), \tag{1.3}$$

where $\sigma = \sigma(x)$, $g^{ij}(x, \dot{x})$ being the contravariant components of the metric tensor of F_n. The space $\bar{F}_n$ with the entities $\bar{F}$, $\bar{g}_{ij}$, etc. is called a conformal Finsler space.

The covariant derivative of a vector $X^i(x, \dot{x})$, depending on the element of support, with respect to x^j in the sense of Cartan is given by [1]

$$X^i_{|j}(x, \dot{x}) \stackrel{\text{def}}{=} (\partial_j X^i) - (\dot{\partial}_m X^i) G^m_j + X^m \Gamma^{*i}_{mj} \; ^{(2)}, \tag{1.4}$$

where

$$G^i_j(x, \dot{x}) \stackrel{\text{def}}{=} G^i_{mj}(x, \dot{x}) \dot{x}^m = \Gamma^{*i}_{mj}(x, \dot{x}) \dot{x}^m . \tag{1.5}$$

The functions $\Gamma^{*i}_{mj}(x, \dot{x})$ and $G^i_{mj}(x, \dot{x})$ are the Cartan's and Berwald's connection coefficients respectively and are homogeneous of degree zero in their directional arguments. We have the following geometric entities in the conformal Finsler space: [3, 4]

$$\bar{G}^i(x, \dot{x}) = G^i(x, \dot{x}) - \sigma_m B^{im}(x, \dot{x}), \tag{1.6}$$

$$\bar{G}^i_j(x, \dot{x}) = G^i_j(x, \dot{x}) - \sigma_m \dot{\partial}_j B^{im}(x, \dot{x}), \tag{1.7}$$

$$\bar{G}^i_{jk}(x, \dot{x}) = G^i_{jk}(x, \dot{x}) - \sigma_m \dot{\partial}_k \dot{\partial}_j B^{im}(x, \dot{x}), \tag{1.8}$$

(*) Nella seduta del 9 marzo 1968.

(1) Numbers in brackets refer to the references at the end of the paper.

(2) $\partial_i \equiv \partial/\partial x^i$, $\dot{\partial}_i \equiv \partial/\partial \dot{x}^i$.

$$(1.9) \qquad \bar{A}^i_{jk}(x, \dot{x}) = e^{\sigma} A^i_{jk}(x, \dot{x}),$$

$$(1.10) \qquad \bar{l}^i(x, \dot{x}) = \bar{e}^{\sigma} l^i(x, \dot{x}), \qquad (1.11) \qquad \bar{l}_i(x, \dot{x}) = e^{\sigma} l_i(x, \dot{x}),$$

$$(1.12) \qquad \bar{\dot{x}}^i = \dot{x}^i, \qquad (1.13) \qquad \bar{C}^i_{jk}(x, \dot{x}) = C^i_{jk}(x, \dot{x}),$$

$$(1.14) \qquad \bar{\Gamma}^{*i}_{jk}(x, \dot{x}) = \Gamma^{*i}_{jk}(x, \dot{x}) + U^i_{jk}(x, \dot{x}),$$

where we have

$$(1.15) \qquad \sigma_m \overset{\text{def}}{=} \partial_m \sigma, \qquad (1.16) \qquad B^{ij}(x, \dot{x}) \overset{\text{def}}{=} \frac{1}{2} F^2 g^{ij} - \dot{x}^i \dot{x}^j,$$

$$(1.17) \qquad l^i(x, \dot{x}) \overset{\text{def}}{=} \dot{x}^i \mid F(x, \dot{x}), \qquad (1.18) \qquad l_i(x, \dot{x}) \overset{\text{def}}{=} g_{ik}(x, \dot{x}) l^k(x, \dot{x}),$$

and

$$(1.19) \qquad A^i_{mj}(x, \dot{x}) \overset{\text{def}}{=} F(x, \dot{x}) C^i_{mj}(x, \dot{x}) \overset{\text{def}}{=} F(x, \dot{x}) g^{ih}(x, \dot{x})$$

$$C_{mhj}(x, \dot{x}) = \frac{1}{2} F(x, \dot{x}) g^{ih}(x, \dot{x}) \dot{\partial}_j g_{mh}(x, \dot{x}),$$

$$(1.20) \qquad U^i_{jk}(x, \dot{x}) \overset{\text{def}}{=} 2\sigma_{(j}\delta^i_{k)} - \sigma_m \{g^{im} g_{jk} - 2 C^i_{r(j}\dot{\partial}_{k)} B^{rm} + g^{ir} C_{jks} \dot{\partial}_r B^{sm}\}.$$

Contracting (1.20) with respect to the indices i and j and using the homogeneity property of the function $C_{ijk}(x, \dot{x})$, we obtain

$$(1.21) \qquad U^r_{rk}(x, \dot{x}) = n\sigma_k + \sigma_m C^n_{rn} \dot{\partial}_k B^{rm}.$$

The functions $B^{ij}(x, \dot{x})$ are homogeneous of degree two in $\dot{x}^{i}$' s.

2. Some entities of the conformal Finsler space.

If we denote the Cartan's covariant derivative given by (1.4) with respect to $\bar{g}_{ij}(x, \dot{x})$ by putting a horizontal bar over the same notation of the covariant derivative, then we have [5]:

$$(2.1) \qquad X^i_{\bar{|}j}(x, \dot{x}) = (\partial_j X^i) - (\dot{\partial}_m X^i) \bar{G}^m_j + X^m \bar{\Gamma}^{*i}_{mj}.$$

With the help of equation (2.1) the covariant derivative of $\bar{l}^i(x, \dot{x})$ in the sense of Cartan is given by

$$(2.2) \qquad \bar{l}^i_{\bar{|}j}(x, \dot{x}) = (\partial_j \bar{l}^i) - (\dot{\partial}_m \bar{l}^i) \bar{G}^m_j + \bar{l}^m \bar{\Gamma}^{*i}_{mj}.$$

Using equations (1.7), (1.10) and (1.14) and the relation $l^i_{|k}(x, \dot{x}) = 0$, in (2.2), we get

$$(2.3) \qquad \bar{l}^i_{\bar{|}j}(x, \dot{x}) = -\bar{e}^{\sigma}\{l^i \sigma_j - (\dot{\partial}_m l^i)(\dot{\partial}_j B^{mn}) \sigma_n + l^m U^i_{mj}\}.$$

Again, we have

$$(2.4) \qquad \bar{l}_{i\bar{|}j}(x, \dot{x}) = (\partial_j \bar{l}_i) - (\dot{\partial}_m \bar{l}_i) \bar{G}^m_j - \bar{l}_m \bar{\Gamma}^{*m}_{ij}.$$

In view of equations (1.7), (1.11) and (1.14) and the relation $l_{i|k}(x, \dot{x}) = 0$, we obtain

$$\bar{l}_{i\overline{|}k}(x, \dot{x}) = e^{\sigma}\{l_i \sigma_k + (\dot{\partial}_m l_i)(\dot{\partial}_k B^{mn})\sigma_n - l_m U^m_{ik}\}. \tag{2.5}$$

Similarly, we obtain the following relations:

$$\bar{F}_{\overline{|}j}(x, \dot{x}) = e^{\sigma}\{F\sigma_j + l_m \sigma_n \dot{\partial}_j B^{mn}\}, \tag{2.6}$$

$$\bar{C}^i_{jk\overline{|}m}(x, \dot{x}) = C^i_{jk|m} + (\dot{\partial}_r C^i_{jk})(\dot{\partial}_m B^{rn})\sigma_n + C^r_{jk} U^i_{rm} - C^i_{rk} U^r_{jm} - C^i_{jr} U^r_{km}, \tag{2.7}$$

and

$$\bar{A}^i_{kh\overline{|}j}(x, \dot{x}) = \tag{2.8}$$

$$= e^{\sigma}[A^i_{kh|j} + A^i_{kh}\sigma_j + (\dot{\partial}_r A^i_{kh})(\dot{\partial}_j B^{rn})\sigma_n + A^r_{kh} U^i_{rj} - A^i_{rh} U^r_{kj} - A^i_{kr} U^r_{hj}].$$

3. Projective invariants in conformal Finsler space.

THEOREM 3.1. If $\bar{l}^i(x, \dot{x})$ and $\bar{l}_i(x, \dot{x})$ are the contravariant and covariant components of the unit tangent vector, in the conformal Finsler space, then we have

$$\bar{B}^i_k(x, \dot{x}) = \bar{e}^{\sigma}\Big[B^i_k(x, \dot{x}) + l^i \sigma_k - (\dot{\partial}_m l^i)(\dot{\partial}_k B^{mn})\sigma_n - l^m U^i_{mk} + \tag{3.1}$$

$$+ \frac{1}{n+1}\{2\delta^i_{(j} U^r_{k)r} + \dot{x}^i \dot{\partial}_j U^r_{rk}\} l^j - \frac{2}{n+1}(\dot{\partial}_h l^i)\dot{x}^r \delta^h_{(r} U^m_{k)m}\Big] \text{(3)},$$

and

$$^*\bar{B}_{ik}(x, \dot{x}) = e^{\sigma}\Big[^*B_{ik}(x, \dot{x}) + l_i \sigma_k + (\dot{\partial}_m l_i)(\dot{\partial}_k B^{mn})\sigma_n - l_m U^m_{ik} + \tag{3.2}$$

$$+ \frac{1}{n+1}\{2\delta^j_{(i} U^r_{k)r} + \dot{x}^j \dot{\partial}_i U^r_{rk}\} l_j + \frac{2}{n+1}(\dot{\partial}_h l_i)\delta^h_{(r} U^m_{k)m}\dot{x}^r\Big].$$

Proof. For a vector $V^i(x, \dot{x})$, depending on the element of support, we have the following projective invariant geometric entities [6]:

$$B^i_k(x, \dot{x}) \stackrel{\text{def}}{=} V^i_{|k} - \frac{1}{n+1}\{2\delta^i_{(j}\Gamma^{*r}_{k)r} + \dot{x}^i \dot{\partial}_j \Gamma^{*r}_{rk}\} V^j + \tag{3.3}$$

$$+ \frac{2}{n+1}(\dot{\partial}_h V^i)\delta^h_{(r}\Gamma^{*m}_{k)m}\dot{x}^r.$$

Under the conformal transformation the above projective invariant entities transform as follows for the unit tangent vector $\bar{l}^i(x, \dot{x})$ of the conformal Finsler space:

$$\bar{B}^i_k(x, \dot{x}) \stackrel{\text{def}}{=} \bar{l}^i_{\overline{|}k} - \frac{1}{n+1}\{2\delta^i_{(j}\bar{\Gamma}^{*r}_{k)r} + \bar{\dot{x}}^i \dot{\partial}_j \bar{\Gamma}^{*r}_{rk}\}\bar{l}^j + \frac{2}{n+1}(\dot{\partial}_h \bar{l}^i)\delta^h_{(r}\bar{\Gamma}^{*m}_{k)m}\bar{\dot{x}}^r. \tag{3.4}$$

(3) We denote the symmetric and skew-symmetric parts of a geometric object A_{ij} with respect to the indices i, j by

$$A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji}) \text{ and } A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji}), \text{ respectively.}$$

Using relations (1.10), (1.12), (1.14) and (2.3) and the fact that the function σ is of points only, we obtain the result (3.1).

Similarly, for a covariant vector $V_i(x, \dot{x})$, depending on the element of support, we have the following projective invariant geometric entities [6]

$$(3.5) \qquad {}^*B_{ik}(x, \dot{x}) \overset{\text{def}}{=} V_{i|k} + \frac{1}{n+1}\{2\delta^j_{(i}\Gamma^{*r}_{k)r} + \dot{x}^j\dot{\partial}_i\Gamma^{*r}_{rk}\}V_j + + \frac{2}{n+1}(\dot{\partial}_h V_i)\delta^h_{(r}\Gamma^{*m}_{k)m}\dot{x}^r.$$

Under the conformal transformation (1.1) the above projective invariant entities transform as follows for the covariant component $\bar{l}_i(x, \dot{x})$ of the conformal Finsler space:

$$(3.6) \qquad {}^*\bar{B}_{ik}(x, \dot{x}) \overset{\text{def}}{=} \bar{l}_{i\bar{|}k} + \frac{1}{n+1}\{2\delta^j_{(i}\bar{\Gamma}^{*r}_{k)r} + \bar{\dot{x}}^j\dot{\partial}_i\bar{\Gamma}^{*r}_{rk}\}\bar{l}_j + + \frac{2}{n+1}(\dot{\partial}_h\bar{l}_i)\delta^h_{(r}\bar{\Gamma}^{*m}_{k)m}\bar{\dot{x}}^r.$$

With the help of equations (1.11), (1.12), (1.14) and (2.5) the above relation gives the result (3.2).

THEOREM 3.2. – If $\bar{l}^i(x, \dot{x})$ and $\bar{l}_i(x, \dot{x})$ are the contravariant and covariant components of the unit tangent vector of the conformal Finsler space, then we have

$$(3.7) \qquad \bar{T}^i(x, \dot{x}) = \bar{e}^{\sigma}[T^i(x, \dot{x}) + 2B^{mn}\sigma_n(\dot{\partial}_m l^i) - \{l^i\sigma_k + l^m U^i_{mk}\}\dot{x}^k + + \frac{2}{n+1} l^{[i}\dot{x}^{r]}U^m_{mr}],$$

and

$$(3.8) \qquad {}^*\bar{T}_i(x, \dot{x}) = e^{\sigma}[T_i(x, \dot{x}) + 2B^{mn}\sigma_n(\dot{\partial}_m l_i) + \{l_i\sigma_k - l_m U^m_{ik}\}\dot{x}^k + + \frac{1}{n+1}\{3l_i U^m_{mr} + l_r U^m_{mi}\}\dot{x}^r].$$

Proof. – We have the following projective invariant entities for a vector $V^i(x, \dot{x})$, depending on the element of support, in Finsler space [6]:

$$(3.9) \qquad T^i(x, \dot{x}) \overset{\text{def}}{=} V^i_{|k}\dot{x}^k + \frac{2}{n+1}V^{[i}\dot{x}^{r]}\Gamma^{*m}_{mr}.$$

Under the conformal change (1.1) the above equation transforms as follows for the contravariant component of the unit tangent vector of the conformal Finsler space:

$$(3.10) \qquad \bar{T}^i(x, \dot{x}) \overset{\text{def}}{=} \bar{l}^i_{\bar{|}k}\bar{\dot{x}}^k + \frac{2}{n+1}\bar{l}^{[i}\bar{\dot{x}}^{r]}\bar{\Gamma}^{*m}_{mr}.$$

Using equations (1.10), (1.12), (1.14) and (2.3) and the Euler's equation for the homogeneous function $B^{ij}(x, \dot{x})$, we obtain the result (3.7).

Again, we have

$$(3.11) \qquad {}^*\mathrm{T}_i(x\,,\dot{x}) \overset{\text{def}}{=} \mathrm{V}_{i|k}\,\dot{x}^k + \frac{1}{n+1}\left\{3\mathrm{V}_i\,\Gamma^{*m}_{mr} + \mathrm{V}_r\,\Gamma^{*m}_{mi}\right\}\dot{x}^r\,,$$

projectively invariant in Finsler space [6].

The projective invariant entities (3.11) transform as follows in the conformal Finsler space for the covariant component of the unit tangent vector $\bar{l}_i(x\,,\dot{x})$:

$$(3.12) \qquad {}^*\bar{\mathrm{T}}_i(x\,,\dot{x}) \overset{\text{def}}{=} \bar{l}_{i\,\bar{|}\,k}\,\bar{\dot{x}}^k + \frac{1}{n+1}\left\{3\,\bar{l}_i\,\bar{\Gamma}^{*m}_{mr} + \bar{l}_r\,\bar{\Gamma}^{*m}_{mi}\right\}\bar{\dot{x}}^r\,.$$

In view of equations (1.11), (1.12), (1.14) and (2.5) and the relation $\dot{\partial}_k\,\mathrm{B}^{ij}(x\,,\dot{x})\,\dot{x}^k = 2\,\mathrm{B}^{ij}(x\,,\dot{x})$, in (3.12) we get (3.8).

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