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On a generalization of the Arghiriade-Dragomir representation of the Moore-Penrose inverse

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Algebra. — *On a generalization of the Arghiriade–Dragomir representation of the Moore-Penrose inverse.* Nota di MIRKO RADIČ, presentata (*) dal Socio B. SEGRE.

SUNTO. — Si risponde a certi problemi di minimo usufruendo della nozione di inversa secondo Moore–Penrose di una matrice rettangolare.

1. Let A be the $m \times n$ matrix whose elements are real numbers and assume rank $A = r$, $A = PQ$, where P and Q are matrices of type $m \times r$, $r \times n$ respectively. The matrix

$$A^{-1} = Q^{-1} P^{-1},$$

where (1)

$$P^{-1} = (P^{\mathrm{T}} P)^{-1} P^{\mathrm{T}} \quad , \quad Q^{-1} = Q^{\mathrm{T}} (QQ^{\mathrm{T}})^{-1},$$

is known as the Moore–Penrose inverse of the rectangular matrix A (see [5] and [6]).

This inverse has a very useful characteristic: If the system $AX = B$ is not consistent, then $(AX - B)^{\mathrm{T}} (AX - B)$ is minimum for $X = A^{-1} B$ (see [7]).

The following question arises: To find the inverse that has the characteristic property

$$v_1^{2k} + \cdots + v_m^{2k} = \text{minimum} \qquad (k = 2, 3, \cdots),$$

where $[v_1, \cdots, v_m]^{\mathrm{T}} = AX - B$?

In this paper we shall give a definition which gives an answer to this question in some specific cases.

2. In [1] Arghiriade and Dragomir have showed that the inverse

$$A^{-1} = A^{\mathrm{T}} (AA^{\mathrm{T}})^{-1}$$

of the matrix A of type $m \times n$ and rank $A = m$ can be represented in the form

$$A^{-1} = \frac{1}{N(A)} \begin{bmatrix} A_{11} \cdots A_{m1} \\ \vdots \\ A_{1n} \cdots A_{mn} \end{bmatrix},$$

where $N(A) = \det(AA^{\mathrm{T}})$ and A_{ij} the " algebraic complement " of a_{ij} is obtained as follows: the determinant of every $m \times m$ submatrix A_i of A in which a_{ij} occurs is multiplied by the algebraic complement of a_{ij} in A_i and then all that is added.

(*) Nella seduta del 9 marzo 1968.

(1) The transpose of A is A^{T}.

For example, the "algebraic complement" of a_{23} in the matrix

(1) $$A = \begin{bmatrix} a_{11}\, a_{12}\, a_{13}\, a_{14} \\ a_{21}\, a_{22}\, a_{23}\, a_{24} \\ a_{31}\, a_{32}\, a_{33}\, a_{34} \end{bmatrix}$$

is

$$A_{23} = \begin{vmatrix} a_{11}\, a_{12}\, a_{13} \\ a_{21}\, a_{22}\, a_{23} \\ a_{31}\, a_{32}\, a_{33} \end{vmatrix} (-1) \begin{vmatrix} a_{11}\, a_{12} \\ a_{31}\, a_{32} \end{vmatrix} + \begin{vmatrix} a_{11}\, a_{13}\, a_{14} \\ a_{21}\, a_{23}\, a_{24} \\ a_{31}\, a_{33}\, a_{34} \end{vmatrix} \begin{vmatrix} a_{11}\, a_{14} \\ a_{31}\, a_{34} \end{vmatrix} + \begin{vmatrix} a_{12}\, a_{13}\, a_{14} \\ a_{22}\, a_{23}\, a_{24} \\ a_{32}\, a_{33}\, a_{34} \end{vmatrix} \begin{vmatrix} a_{12}\, a_{14} \\ a_{32}\, a_{34} \end{vmatrix} .$$

In the same way the inverse

$$A^{-1} = (A^{\mathrm{T}} A)^{-1} A^{\mathrm{T}} \ (m > n)$$

can be represented.

3. Let us form the inverse of an $m \times n$ matrix A, whose rank is m, as follows:

$$A^{-1} = \frac{1}{N(A)} \begin{bmatrix} A_{11} \cdots A_{m1} \\ \vdots \qquad \vdots \\ A_{1n} \cdots A_{mn} \end{bmatrix} ,$$

where

$$N(A) = \sum_{i=1}^{\binom{n}{m}} |A_i|^{\frac{1}{2k-1}} \, |A_i| \qquad (k \text{ is any positive integer}),$$

$$A_{ij} = \sum (|A_i|^{\frac{1}{2k-1}} \times \text{algebraic complement of } a_{ij} \text{ in } |A_i|).$$

For example, "the algebraic complement" of a_{23} in the matrix (1) is

$$A_{23} = \begin{vmatrix} a_{11}\, a_{12}\, a_{13} \\ a_{21}\, a_{22}\, a_{23} \\ a_{31}\, a_{32}\, a_{33} \end{vmatrix}^{\frac{1}{s}} (-1) \begin{vmatrix} a_{11}\, a_{12} \\ a_{31}\, a_{32} \end{vmatrix} + \begin{vmatrix} a_{11}\, a_{13}\, a_{14} \\ a_{21}\, a_{23}\, a_{24} \\ a_{31}\, a_{33}\, a_{34} \end{vmatrix}^{\frac{1}{s}} \begin{vmatrix} a_{11}\, a_{14} \\ a_{31}\, a_{34} \end{vmatrix} + \begin{vmatrix} a_{12}\, a_{13}\, a_{14} \\ a_{22}\, a_{23}\, a_{24} \\ a_{32}\, a_{33}\, a_{34} \end{vmatrix}^{\frac{1}{s}} \begin{vmatrix} a_{12}\, a_{14} \\ a_{32}\, a_{34} \end{vmatrix} ,$$

where $s = 2k - 1$.

Similarly we proceed for the case when $m > n$ and rank $A = n$.

We shall now state some characteristics of the above definition. It is easy to see that

$$(MA)^{-1} = A^{-1} M^{-1}$$

where A is an $m \times n$ matrix, rank $A = m$, M is a square regular matrix. If $m > n$, rank $A = n$ then

$$(AM)^{-1} = M^{-1} A^{-1}.$$

If $A = PQ = RS$, where P and R, Q and S are matrices of type $m \times r$, $r \times n$ respectively, rank $A = r$, then

$$Q^{-1} P^{-1} = S^{-1} R^{-1},$$

because:

$$R = PQS^{-1} = PM \quad , \quad S = R^{-1} PQ = NQ \quad , \quad MN = I,$$

$$S^{-1} R^{-1} = Q^{-1} N^{-1} M^{-1} P^{-1} = Q^{-1} P^{-1}.$$

If the system

$$a_{i1} x_1 + \cdots + a_{in} x_n = b_i \qquad (i = 1, \cdots, m)$$

is not consistent and $m = n + 1$, the rank of the matrix $A = [a_{ik}]$ being n, then

$$v_1^{2k} + \cdots + v_m^{2k} = \text{minimum}$$

for $X = A^{-1} B$.

In order to prove it, let us put

$$y_i = a_{i1} x_1 + \cdots + a_{in} x_n \qquad (i = 1, \cdots, m),$$

$$D_0 = \begin{vmatrix} a_{11} \cdots a_{1n} \\ \vdots \qquad \vdots \\ a_{n1} \cdots a_{nn} \end{vmatrix}, \quad D_i = \begin{vmatrix} a_{11} \cdots a_{1n} \\ \cdots\cdots\cdots \\ a_{m1} \cdots a_{mn} \\ \cdots\cdots\cdots \\ a_{n1} \cdots a_{nn} \end{vmatrix} \text{(ith row)} \quad (i = 1, \cdots, n).$$

If, let us say, $D_0 \neq 0$, then the system

$$\frac{\partial}{\partial x_i} [(y_1 - b_1)^{2k} + \cdots + (y_m - b_m)^{2k}] = 0 \qquad (i = 1, \cdots, n),$$

or

$$a_{11} (y_1 - b_1)^{2k-1} + \cdots + a_{m1} (y_m - b_m)^{2k-1} = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{1n} (y_1 - b_1)^{2k-1} + \cdots + a_{mn} (y_m - b_m)^{2k-1} = 0,$$

can be written in the form

$$y_i - b_i = -\left(\frac{D_i}{D_0}\right)^{\frac{1}{2k-1}} (y_m - b_m) \qquad (i = 1, \cdots, n).$$

It is not difficult to see that $X = A^{-1} B$ is the solution of this system.

However, if $m \neq n+1$ a similar result is not generally true. For example, for the system

$$x_1 - x_2 = 4$$

$$x_1 - x_2 = 0$$

$$-x_1 + x_2 = 1$$

$$x_1 + 7 x_2 = 2$$

we have (for the case when $k = 2$)

$$X = A^{-1} B = \frac{1}{24} \begin{bmatrix} 7 & 7 & -7 & 3 \\ -1 & -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 9 \\ 1 \end{bmatrix}.$$

However, this X is not a solution of the system:

$$(x_1 - x_2 - 4)^3 + (x_1 - x_2)^3 - (-x_1 + x_2 - 1)^3 + (x_1 + 7x_2 - 2)^3 = 0,$$

$$-(x_1 - x_2 - 4)^3 - (x_1 - x_2)^3 + (-x_1 + x_2 - 1)^3 + 7(x_1 + 7x_2 - 2)^3 = 0.$$

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