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# Rendiconti

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## On the first boundary value problem for equations of elliptic type degenerating on the boundary

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# RENDICONTI

DELLE SEDUTE

## DELLA ACCADEMIA NAZIONALE DEI LINCEI

**Classe di Scienze fisiche, matematiche e naturali**

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## SEZIONE I

**(Matematica, meccanica, astronomia, geodesia e geofisica)**

**Analisi matematica.** — *On the first boundary value problem for equations of elliptic type degenerating on the boundary.* Nota di S. A. TERSENOV, presentata (*) dal Socio straniero S. L. SOBOLEV.

RIASSUNTO. — In questa Nota viene considerato il primo problema al contorno per l'equazione del tipo ellittico

$$a^{ij} u_{x_i x_j} + b^i u_{x_i} + cu = 0 \quad (c \leq 0) \tag{1}$$

in un dominio $G \in A^{(2,\alpha)}$. L'equazione (1) può degenerare sulla frontiera S del dominio G. Si suppone che i coefficienti dell'equazione (1) appartengano alla classe $C^{(\alpha)}(G)$ e siano limitati in $\overline{G}$. Si costruisce un insieme $S^{(1)}$ di S tale che esista una soluzione regolare $u$ dell'equazione (1) limitata in $\overline{G}$ soddisfacente alla condizione

$$u = f \quad \text{su } S^{(1)}.$$

Si dimostra che se una soluzione dell'equazione (1) è limitata in $\overline{G}$ e identicamente nulla su $S^{(1)}$ essa è identicamente nulla in G.

In paper [1] it was shown for a class of elliptic equations degenerating on the boundary that for the solvability of the Dirichlet problem within a class of bounded functions it is necessary (depending on coefficients) to eliminate of the boundary conditions a part of the boundary in which the equations is degenerated.

In numerous papers [2–10] some generalizations of the results obtained were given as well as conditions of solvability of the first boundary value problem

(*) Nella seduta del 10 febbraio 1968.

in a classical formulation for the second order degenerated equations of elliptic type.

Consider equation:

$$(1) \qquad L(u) \equiv a^{ij} u_{x_i x_j} + b^i u_{x_i} + cu = 0$$

of elliptic type in the domain $G \in A^{(2,\alpha)}$ (see [16]) of the $n$–dimensional space with the boundary S which can degenerate into $G + S$. In papers [11–15] for $c < 0$ in $G + S$ and $a^{ij} \in C^{(2,\alpha)}(G + S)$, $b^i \in C^{(1,\alpha)}(G + S)$ and $c \in C^{(\alpha)}(G + S)$ in terms of coefficients is indicated a set of the points of S to be eliminated of the boundary conditions so that there would always exist only one generalized solution $u$ of Eq. (1). It may be easily shown that these results are generally invalid if the coefficients belong to a wider class of functions remaining continuous in $G + S$.

For example, for equation

$$yu_{yy} + u_{xx} + \left(1 + \frac{k}{\log y}\right) u_y - u = 0 \; (y > 0)$$

the Dirichlet problem with the data on the whole boundary of the domain based on the line of degeneration of $y = 0$ is always solvable for $k > 1$ though $b = 0$ for $y = 0$ (see [12]). We also note that if $c \leq 0$ in G, and Eq. (1) degenerates in G, then there generally does not hold the maximum principle even for regular solutions. For example, function $u = (1 - r^2)^2$ in the circle $r^2 = x^2 + y^2 < 1$ is a regular solution of the equation

$$(1 - r^2)^2 r^2 \Delta u - 2 (1 - r^2)^2 (xu_x + yu_y) - 8 r^y u = 0$$

and $u = 0$ for $r = 1$.

I. Let Eq. (1) be of elliptic type in a bounded domain $G \in A^{(2,\alpha)}$ which can degenerate just on S boundary G. Assume that the coefficients of Eq. (1) belong to the class $C^{(\alpha)}(G)$ and that they are restricted in $G + S$, moreover $c \leq 0$ in G. The supplementary condition placed on the coefficient will be indicated below.

Let $G_0 \subset G$ be a boundary strip of sufficient thinness. We denote by $G_k \subset G_0$ $(k = 1, \cdots, N)$ a finite set of domains covering $G_0$, and let

$$(2) \qquad (s_e = s_e(x_1, \cdots, x_n), \; t = t(x_1, \cdots, x_n), \; e = 1, \cdots, n - 1)$$

be a system of functions which are continuously twice differentiated on $\overline{G}_k$ in Hölder sense with a Jacobian which differs from zero and maps the domain $G_k$ onto the parallelepiped $\Pi_k$:

$$(3) \qquad -M \leq s_e - s_e^0 \leq M, \; 0 \leq t \leq R, \; (e = 1, \cdots, n - 1)$$

with the bases on the planes $t = 0$, $t = R$; while $t(x) = 0$ for all $x \in S_k$, where $S_k$ is a part of the boundary $G_k$ belonging to S. We note that index "$k$" is omitted for the sake of simplicity in functions (2). In the domain $G_k$

using the variables $s_1, \cdots, s_{n-1}, t$, Eq. (1) can be written as:

$$(4) \qquad A\frac{\partial^2 u}{\partial t^2} + \sum_{i,j=1}^{n-1} A_{ij}\frac{\partial^2 u}{\partial s_i \partial s_j} + \sum_{i=1}^{n-1} B_i \frac{\partial^2 u}{\partial t \partial s_i} + a\frac{\partial u}{\partial t} + \sum_{j=1}^{n-1} b_i \frac{\partial u}{\partial s_i} + cu = 0.$$

The coefficients of Eq. (4) satisfy the same conditions as Eq. (1).

We denote by $s$ a point with the coordinates $(s_1, \cdots, s_{n-1})$. From the ellipticity of Eq. (1) it follows that $A(s, t) > 0$ for $t > 0$.

We consider the function:

$$v(p) = v(s, t) = \int_t^R d\tau \int_\tau^R \frac{1}{A(s, \tau_1)} \left( exp \int_\tau^{\tau_1} \frac{a(s, \tau_2)}{A(s, \tau_2)} d\tau_2 \right) d\tau_1$$

where $p \in \Pi_k$.

Evidently, $v(p)$ for $t > 0$ satisfies Hölder condition while over $t$ it is continuously twice differentiated and satisfies equation:

$$\frac{\partial^2 v}{\partial t^2} + \frac{a(s, t)}{A(s, t)} \frac{\partial v}{\partial t} = \frac{1}{A(s, t)}.$$

We extend functions $A(s, t)$ and $a(s, t)$ from the parallelepiped (3) to the whole strip:

$$-\infty < s_e < \infty, 0 < t \le R, (e = 1, \cdots, n-1) \qquad \text{periodically.}$$

Then function $v(p)$ will be periodically extended to the strip:

$$-\infty < s_e < \infty, 0 < t \le R, (e = 1, \cdots, n-1).$$

We denote by $Q_k$ the base $t = 0$ of the parallelepiped $\Pi_k$.

Let $Q_k^{(1)}$ be a set of points $q \in Q_k$ in which

$$(5) \qquad \overline{\lim_{p \to q}}\, v(p) < \infty,$$

$Q_k^{(2)}$ be a set of points $q \in Q_k$ in which

$$(6) \qquad \underline{\lim_{p \to q}}\, v(p) < \infty, \ \overline{\lim_{p \to q}}\, v(p) = \infty,$$

and $Q_k^{(3)}$ be a set of points $q \in Q_k$ in which

$$(7) \qquad \underline{\lim_{p \to q}}\, v(p) = \infty.$$

We denote by $S_k^{(i)}$ the images $Q_k^{(i)}$ on a part of the surface $S_k$, respectively.

Let us introduce the notations:

$$S^{(1)} = \bigcup_k S_k^{(1)} \quad , \quad S^{(2)} = \bigcup_k S_k^{(2)} \quad , \quad S^{(3)} = \bigcup_k S_k^{(3)}.$$

LEMMA 1.

1) *The sets* $S^{(i)}$ *are not dependent of the transform* (2),

2) $S^{(1)}$ *is an open set*, $S^{(2)}$ *is a set without interior points*,

3) $\overline{S^{(1)}} \supseteq S^{(1)} \cup S^{(2)}$, $\overline{S^{(3)}} = S^{(3)} \cup S^{(2)}$.

II. By a regular solution of Eq. (1) we understand a continuous twice differentiated solution. Below we shall use the notations of a superfunction and a subfunction for the Eq. (1), whose definitions and the properties can be found in [17].

LEMMA 2. *Let* $S_0 \subset S$ *be a closed set. If in the domain* G *there exists the superfunction* $w$ *satisfying conditions*: $w > 0$ *in* $G + S$, *and* $w \to \infty$ *is uniform in approaching* $S_0$, *then every regular in* G *solution* $u$ *of Eq.* (1) *which satisfies conditions*:

$$\lim_{x \to y \in S - S_0} u(x) = 0 \,, \quad \lim_{x \to y \in S_0} \frac{u(x)}{w(x)} = 0 \tag{8}$$

is identical zero in G.

Indeed, function $\varphi = \varepsilon w \pm u$ is a superfunction for every $\varepsilon > 0$ and $\varphi > 0$ on S. Therefore, $\varphi > 0$ in G. Hence in virtue of the arbitrariness of $\varepsilon$ it follows that $u \equiv 0$.

LEMMA 3. (Maximum principle). *Let conditions of the Lemma 2 be fulfilled, and let for the regular in* G *solution* $u$ *both positive the upper boundary* M *and the negative lower boundary* $m$ *on the* $\overline{S - S_0}$ *be bounded.* Then, if

$$\lim_{x \to y \in S_0} \frac{u(x)}{w(x)} = 0$$

then $u$ is bounded in $G + S$, and

$$m \le u \le M \quad \text{in} \quad G. \tag{9}$$

Indeed, functions $\varepsilon w + u - m$, $\varepsilon w + M - u$ for every $\varepsilon > 0$ are superfunctions, and $\varepsilon w + u - m \ge 0$, $\varepsilon w + M - u \ge 0$ on S.

Therefore,

$$\varepsilon w + u - m \ge 0 \,, \varepsilon w + M - u \ge 0 \quad \text{in} \quad G.$$

In virtue of the arbitrariness of $\varepsilon$ follows (9). It is evident that these Lemmas imply the following maximum principle (see, also [12], [13]):

If conditions of the Lemma 3 are fulfilled, then

$$|u(x)| \le \sup_{y \in S - S_0} \overline{\lim_{x \to :}} |u(x)| \,. \tag{10}$$

It may be easily seen that if $y(s) \in \gamma \subset S^{(1)}$, where $\gamma$ is in some neighbourhood of the point $y(s)$, then there exists the function:

$$\omega(t) = \max_{s \in \gamma} \int_0^t d\tau \int_\tau^R \frac{1}{A(s, \tau_1)} \left( \exp \int_\tau^{\tau_1} \frac{a(s, \tau_2)}{A(s, \tau_2)} d\tau_2 \right) d\tau_1 \tag{11}$$

and if $y(s) \in \gamma \subset S^{(3)}$ ($y(s)$—being an interior point of $S^{(3)}$), then function:

$$v(t) = \min_{s \in \gamma} \int_t^R d\tau \int_\tau^R \frac{1}{A(s, \tau_1)} \left( \exp \int_\tau^{\tau_1} \frac{a(s, \tau_2)}{A(s, \tau_2)} d\tau_2 \right) d\tau_1 \tag{12}$$

has the property: $v(t) \to \infty$ as $t \to 0$.

Now let us assume that the coefficients of Eq. (1), in addition, satisfy condition: at least for one transform of the form (2), and for a single value of the index " $k$ " there exist such $N > 0$ that at every point $y \in S^{(3)}$ there is fulfilled one of the conditions:

$$\overline{\lim_{x \to y}}\, c(x) < 0, \tag{13}$$

or

$$\underline{\lim_{x \to y}}\, [N A_{kk}(x) + b_k(x)] > 0. \tag{14}$$

Let $f$ be a continuous function on $G + S$. Then there holds the following.

THEOREM. 1) *There always exists a regular solution* $u$ *of Eq.* (1) *in* G *bounded in* $G + S$ *which satisfies condition*:

$$u = f \quad \text{on} \quad S^{(1)}, \tag{15}$$

2) *if* $u$ *is a solution of Eq.* (1) *bounded in* $G + S$, *and* $u = 0$ *on* $\overline{S^{(1)}}$, *then* $u \equiv 0$ *in* G.

*Remark.* If $S^{(3)}$ is a closed set without the isolated points, then the sentence 2) is valid also when $u = 0$ on $S^{(1)}$. It is easily seen that there always exists a generalized solution in Winer sense of the problem (1–15). If $y(s_1^0, \dots, s_{n-1}^0) = y(s^0) \in \gamma \subset S^{(1)}$ and there exists integral:

$$\Omega(t) = \int_0^t d\tau \int_\tau^R \max_{s \in \gamma} \frac{1}{A(s, \tau_1)} \left( \exp \int_\tau^{\tau_1} \max_{s \in \gamma} \frac{a(s, \tau_2)}{A(s, \tau_2)}\, d\tau_2 \right) d\tau_1$$

then at the points $y(s^0)$ the so-called local barrier is function:

$$\varphi_0(s, t) = K\Omega(t) + \sum_{i=1}^{n-1} (s_i - s_i^0)^2 ,$$

where $K > 0$ is sufficiently large value, and if $\Omega(t) = \infty$, then function:

$$\varphi(s, t) = \sum_{i=1}^{n-1} (s_i - s_i^0)^2 + K\omega(t)$$

in some neighbourhood of the points $y(s^0)$ be a superfunction for the sufficiently large K ; $\omega(t)$ is the function given by formula (11). The first part of the assumption has been proved.

Let $u = 0$ on $\overline{S^{(1)}}$, and it attains its positive upper boundary M at the interior point $y \in \gamma_0 \subset S^{(3)}$, where $\gamma_0$ is an open connected set whose boundary consists of the points $S^{(2)}$.

Let $y \in \gamma \subset \gamma_0$. If integral

$$v_0(t) = \int_t^R d\tau \int_\tau^R \min_{s \in \gamma} \frac{1}{A(s, \tau_1)} \left( \exp \int_\tau^{\tau_1} \min_{s \in \gamma} \frac{a(s, \tau_2)}{A(s, \tau_2)}\, d\tau_2 \right) d\tau_1$$

has the property: $v_0(t) \to \infty$ as $t \to 0$, then in virtue of conditions (13) and (14), the constants $M_1 > 0$, $N_1 > 0$, $m > 0$ can be chosen so that the function:

$$w_0(s, t) = v_0(t) + M_1 [N_1 - (s_k + s_k^0)^m], \quad (s_k + s_k^0) > 1$$

would be a superfunction, which satisfies conditions of the Lemma 3 in some domain $D \subset G_0$ based on the surface $\gamma \subset S^{(3)}$.

If $v_0(0) < \infty$, then the superfunction in the domain $D \subset G_0$ will be the function:

$$w(s, t) = v(t) + M_1 [N_1 - (s_k + s_k^0)^m]$$

where $v(t)$ is given by formula (12).

In virtue of the properties of the superfunction and the fact that $u$ cannot attain its positive upper boundary inside G, it follows that the solution $u$ attains its positive upper boundary on the boundary of the surface $\gamma$. Using the arbitrariness $\gamma$, one can easily show that there exists a sequence of the points $y^{(m)} \in S^{(3)}$, $y^{(m)} \to y \in S^{(2)}$ in each of which

$$\overline{\lim_{x \to y^{(m)}}}\, u(x) = M .$$

Hence it follows that for every $\varepsilon > 0$ it is always possible to construct such a sequence of the points $x^{(m)} \in G$ that $\lim x^{(m)} = y \in S^{(2)}$, and $u(x^{(m)}) > M - \varepsilon$. Since $\lim u(x^{(m)}) = u(y) = 0$, then we have $M < \varepsilon$, i.e. $M = 0$. In analogy, one can show that the negative lower boundary is equal to zero.

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