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Variational formulation for linear equations of mathematical physics

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Fisica matematica. — *Variational formulation for linear equations of mathematical physics* (*). Nota di ENZO TONTI, presentata (**) dal Socio B. FINZI.

RIASSUNTO. — Sono presentate condizioni necessarie e sufficienti perchè uno o più sistemi di equazioni differenziali lineari siano deducibili dalla stazionarietà di un funzionale. Queste condizioni permettono una sistematica rassegna dei principi variazionali nella fisica matematica, specialmente nelle teorie lineari di campo.

1. INTRODUCTION.

The discovery of variational principles in physics has long been only a matter of chance, as is shown by the fact that sometimes between the discovery of two principles, one of which is the hamiltonian form of the other, a half or even a whole century elapses (1). Later some order was introduced by the discovery that self-adjoint problems admit a variational formulation. More generally the operator can be symmetric, as may be seen in the wonderful book by Mikhlin [1]. The symmetry of the operator is, for a differential operator, a weaker requirement than that of being self-adjoint [2]. But these conditions are only sufficient and not necessary for deducing a system of equations from a functional. In fact, as an example, in mechanics from Hamilton's principle follow Lagrange's equations of motion to which initial conditions are associated: these conditions do not make the operator symmetric and thus Hamilton's principle does not enter the preceding criteria.

We show in this paper the *necessary* condition for an equation to admit variational formulation and various kinds of sufficient conditions. With these conditions a systematic inspection of all known field theories permits to find again the variational principles already known and to find the new ones that complete, in a well defined sense, the number of possible variational principles in each of these theories [6], [7].

2. MATHEMATICAL PRELIMINARIES.

Let us consider systems of linear differential equations, with partial or total derivatives; as a particular case a single equation. The set of derivation symbols which form such a system is called *formal differential operator* [2],

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(1) This is the case of the principles of elastostatics and of electromagnetism: [6], [7].

and in the sequel we shall indicate it with an italic capital letter. So in the following systems of equations [2]:

$$\text{(1)}\qquad \Delta f = g \qquad \nabla_k u^k = f \qquad -\mu\Delta\vec{v} - (\lambda+\mu)\nabla\nabla\cdot\vec{v} = \vec{f}$$

$$\varepsilon^{\alpha\beta\gamma\varrho}\,F_{\alpha\beta/\gamma} = 0 \qquad \frac{1}{2}(\varphi_{\alpha/\beta} - \varphi_{\beta/\alpha}) = F_{\alpha\beta} \qquad \varepsilon^{hrl}\,\varepsilon^{ksm}\,\nabla_r\nabla_s e_{lm} = 0$$

the formal differential operators are:

$$\text{(2)}\qquad \mathcal{E} = \Delta \qquad \mathcal{S} = \nabla_k \qquad \mathcal{A} = -\mu\Delta - (\lambda+\mu)\nabla\nabla\cdot$$

$$\mathcal{B} = \varepsilon^{\alpha\beta\gamma\varrho}\,\nabla_\gamma \qquad \mathcal{R} = \frac{1}{2}(g_{\alpha\gamma}\nabla_\beta - g_{\beta\gamma}\nabla_\alpha) \qquad \mathcal{C} = \varepsilon^{hrl}\,\varepsilon^{ksm}\,\nabla_r\nabla_s\,.$$

Each formal differential operator maps a tensor into another, in general of different order. Let m denote the order of the tensor on which the formal operator works, n that of the transformed tensor.

Let H_m denote the Hilbert space formed by the tensors of order m, H_n that of the tensors of order n. As a scalar product of two elements $\underset{1}{h}$ and $\underset{2}{h}$ of the Hilbert space H_m let us take the following:

$$\text{(3)}\qquad \int_\Omega w\,\underset{1}{h}\,\underset{2}{h}\,d\Omega$$

as w is a symmetric and positive definite tensor called "weight" tensor. So in elasticity, if e_{hk} is the strain tensor, $p^{hk} = c^{hkrs}\,e_{rs}$ the stress tensor, where c^{hkrs} is Hooke's tensor, the following scalar product is useful [1]:

$$\text{(4)}\qquad \int_\Omega C^{hkrs}\,\underset{1}{e_{hk}}\,\underset{2}{e_{rs}}\,d\Omega = \int_\Omega \underset{1}{p^{hk}}\,\underset{2}{e_{hk}}\,d\Omega\,.$$

In electrostatics if E_k is the electric field vector, $D^h = \varepsilon^{hk}E_k$ the electric induction vector and ε^{hk} the dielectric permeability tensor, the following scalar product is useful:

$$\text{(5)}\qquad \int_\Omega \varepsilon^{hk}\,\underset{1}{E_h}\,\underset{2}{E_k}\,d\Omega = \int_\Omega \underset{1}{D^k}\,\underset{1}{E_k}\,d\Omega\,.$$

A formal differential operator $\mathcal{C}$ cannot operate on all the elements h of the space H_m but only on those which have sufficient derivability requirements. Let us consider only those tensors that have derivatives of all orders: these conditions can be considered weakened, but for our purposes this enlargement is not essential. U_m denotes the subspace formed by the tensors of order m satisfying these derivability requirements.

(2) We make use of tensor notations and speak of tensors because they are the most common sets of functions in mathematical physics, but the tensor nature is inessential for what follows: every set of functions, of non tensorial nature, as Cristoffel's symbols or spinors can be used.

The tensors transformed by $\mathcal{T}$ do not fill all H_n but go into the subspace V_n of all the tensors with derivatives of all orders. U_m represents the *ambient space* for the formal differential operator $\mathcal{T}$. We call *formal adjoint* of a formal differential operator $\mathcal{T}$ that formal operator $\tilde{\mathcal{T}}$ which realizes the condition:

$$\int_\Omega v\,\mathcal{T}u\,\mathrm{d}\Omega = \int_\Omega u\,\tilde{\mathcal{T}}v\,\mathrm{d}\Omega + \{\text{boundary terms}\} \tag{6}$$

where if u is a tensor with some symmetry, i.e. symmetric or skew-symmetric, we must make the integration by parts and consider the symmetrical or skew-symmetrical part of the so obtained tensor.

Thus the formal operator $\mathcal{R}$ in (2) is the adjoint of the formal operator $-\nabla_\alpha$ applied to the skew-symmetric tensor $F^{\alpha\beta}$. If $\mathcal{T} = \tilde{\mathcal{T}}$ the formal differential operator is said to be *formally symmetric* and for it the following condition holds:

$$\int_\Omega v\,\mathcal{S}u\,\mathrm{d}\Omega = \int_\Omega u\,\mathcal{S}v\,\mathrm{d}\Omega + \{\text{boundary terms}\}. \tag{7}$$

We note that the symmetry of a formal operator in the case of systems of differential equations can be destroyed by merely changing the order of the equations or making linear combinations with constants or known functions [4]. In particular every ordinary differential equation of second order can always be transformed in order to have a formal symmetric operator, multiplying it by a suitable function.

A formally symmetric operator generates necessarily functions of the same tensorial order than those on which it operates and therefore transforms elements of the space U_m into elements of the same space.

3. Variational formulation of a system of differential equations.

THEOREM I.—To deduce a system of linear differential equations:

$$\mathcal{S}u = 0 \tag{8}$$

from the stationarity of a functional it is necessary that $\mathcal{S}$ be *formally symmetric*, that is:

$$\mathcal{S} = \tilde{\mathcal{S}}. \tag{9}$$

Proof: if a system of linear differential equations can be deduced from a functional, its lagrangian can be supposed to be quadratic (3).

(3) In fact if a system of linear differential equations follows from a lagrangian it can also contain some non quadratic terms but, as can be shown, this term is a divergence so that it can be confined, by the divergence theorem, in the boundary integral and so does not affect the resulting equation.

If u denotes a tensor, the most general quadratic lagrangian is:

$$\text{(10)} \qquad \mathrm{L}(u) = \frac{1}{2}\,\alpha uu + \frac{1}{2}\,\beta^h(\nabla_h u)\,u + \frac{1}{2}\,\gamma^{hk}(\nabla_h u)(\nabla_k u) + $$
$$+ \frac{1}{2}\,\sigma^{hkl}(\nabla_h\nabla_k u)(\nabla_l u) + \frac{1}{2}\,\tau^{hkl}(\nabla_h\nabla_k\nabla_l u)\,u + \cdots$$

which has the form:

$$\text{(11)} \qquad \mathrm{L}(u) = \Sigma_s \frac{1}{2}\,w_s(\mathfrak{R}_s u)(\mathfrak{T}_s u)$$

where w_s are tensors, $\mathfrak{R}_s$ and $\mathfrak{T}_s$ are formal differential operators or the identity operators possibly. The functional is:

$$\text{(12)} \qquad \mathfrak{I}[u] = \int_\Omega \Sigma_s \frac{1}{2}\,w_s(\mathfrak{R}_s u)(\mathfrak{T}_s u)\,\mathrm{d}\Omega$$

and then:

$$\text{(13)} \qquad \delta\mathfrak{I}[u] = \int_\Omega \Sigma_s \frac{1}{2}\,[w_s(\mathfrak{R}_s \delta u)(\mathfrak{T}_s u) + w_s(\mathfrak{R}_s u)(\mathfrak{T}_s \delta u)]\,\mathrm{d}\Omega =$$
$$= \int_\Omega \Sigma_s \delta u \frac{1}{2}\,[(\tilde{\mathfrak{R}}_s w_s \mathfrak{T}_s u) + (\tilde{\mathfrak{T}}_s w_s \mathfrak{R}_s u)]\,\mathrm{d}\Omega + \{\text{boundary terms}\}.$$

If we put $\delta\mathfrak{I}[u] = 0$ and $\delta u = 0$ on the boundary with its derivatives we obtain the equation:

$$\text{(14)} \qquad \Sigma_s \frac{1}{2}\,[\tilde{\mathfrak{R}}_s w_s \mathfrak{T}_s + \tilde{\mathfrak{T}}_s w_s \mathfrak{R}_s]\,u = 0\,.$$

The operator appearing in it is formally symmetric. In fact

$$\text{(15)} \qquad \int_\Omega v\left[\Sigma_s \frac{1}{2}\,(\tilde{\mathfrak{R}}_s w_s \mathfrak{T} + \tilde{\mathfrak{T}}_s w_s \mathfrak{R}_s)\right] u\,\mathrm{d}\Omega =$$
$$= \int_\Omega u\left[\Sigma_s \frac{1}{2}\,(\tilde{\tilde{\mathfrak{T}}}_s w_s \mathfrak{R}_s + \tilde{\tilde{\mathfrak{R}}}_s w_s \mathfrak{T}_s)\right] v\,\mathrm{d}\Omega + \{\text{boundary terms}\}.$$

We enumerate here a few examples: the equations

$$\text{(16)} \quad \left\{ \begin{array}{ll} [\varepsilon^{hrl}\,\nabla_l]\,v_r = f^h & (\mathrm{rot}\ \vec{v} = \vec{f}) \\ [-\nabla_h]\,\mu^{hk}\,[\nabla_k]\,u = f & \text{scalar field equation} \\ [-\varepsilon^{hrl}\,\varepsilon^{ksm}\,\nabla_r\,\nabla_s]\,e_{lm} = 0 & \text{Saint Venant compatibility conditions} \\ [\nabla_\alpha]\,\mathrm{C}^{\alpha\beta\gamma\varrho}\left[\frac{1}{2}\,g_{\gamma\nu}\nabla_\varrho - \frac{1}{2}\,g_{\varrho\nu}\nabla_\gamma\right]\varphi^\nu = \mathfrak{I}^\beta & \text{electromagnetic wave equation} \end{array} \right.$$

have their operators formally symmetric and their corresponding functionals are:

$$
(17)\begin{cases}
\mathfrak{J}_1[v] = \int_\Omega \left(\frac{1}{2}\, v_h\, \varepsilon^{hrl} \nabla_l v_r - v_k f^k\right) d\Omega \equiv \int_\Omega \left(\frac{1}{2}\, \vec{v} \cdot \mathrm{rot}\, \vec{v} - \vec{v} \cdot \vec{f}\right) d\Omega \\
\mathfrak{J}_2[u] = \int_\Omega \left(\frac{1}{2}\, \mu^{hk}\, \nabla_h u\, \nabla_k u - uf\right) d\Omega & \text{generalized Dirichlet principle} \\
\mathfrak{J}_3[e] = \int_\Omega \left(\frac{1}{2}\, \varepsilon^{hrl}\, \varepsilon^{ksm}\, \nabla_r e_{lm} \nabla_s e_{hk}\right) d\Omega & \text{see [6]} \\
\mathfrak{J}_4[\varphi] = \int_\Omega \left(\frac{1}{2}\, C^{\alpha\beta\gamma\varrho}\, \nabla_\alpha \varphi_\beta\, \nabla_\gamma \varphi_\varrho - \mathfrak{J}^\beta \varphi_\beta\right) d\Omega & \text{classic action principle of electromagnetism: see [7].}
\end{cases}
$$

This theorem permits us to obtain the decomposition of a formally symmetric operator $\mathfrak{S}$ as follows:

$$
(18) \qquad \mathfrak{S} = \Sigma_k \frac{1}{2} \left[\widetilde{\mathfrak{R}}_s\, w_s\, \mathfrak{T}_s + \widetilde{\mathfrak{T}}_s\, w_s\, \mathfrak{R}_s\right]
$$

where $\mathfrak{R}_s$ and $\mathfrak{T}_s$ are formal operators, w_s a symmetric tensor, Σ_k means that $\mathfrak{S}$ can be broken down into the sum of terms indicated between brackets. The proof is as follows: from the fact that $\mathfrak{S}$ is formally symmetric we can obtain a functional containing necessarily a quadratic lagrangian. But the most general quadratic lagrangian is given by (11) and the formal operator that we obtain from it is of the kind given by (14): thus the breakdown is demonstrated. In particular if $\mathfrak{S}$ is an even order formal operator it is easy to see that it admits the decomposition:

$$
(19) \qquad \mathfrak{S} = \Sigma_s\, \widetilde{\mathfrak{R}}_s\, w_s\, \mathfrak{R}_s\,.
$$

We give some examples:

$$
(20)\begin{cases}
\varepsilon^{hrl} \nabla_l = \frac{1}{2} \{[\mathrm{I}]\, \varepsilon^{hrl}\, [\nabla_l] + [-\nabla_l]\, \varepsilon^{hrl} [\mathrm{I}]\} \\
-\Delta = [-\nabla_k]\, g^{hk}\, [\nabla_h] \\
-\varepsilon^{hrl}\, \varepsilon^{ksm}\, \nabla_r \nabla_s = [-\nabla_r]\, \varepsilon^{hrl}\, \varepsilon^{ksm}\, [\nabla_s] \\
-\mu\Delta - (\mu + \lambda)\, \nabla\nabla\cdot = [\nabla\cdot]\, \mu\, [-\nabla] + [-\nabla]\, (\lambda + \mu)\, [\nabla\cdot] \\
\Delta + \omega^2 = [-\nabla_k]\, g^{hk}\, [\nabla_h] + [\mathrm{I}]\, \omega^2\, [\mathrm{I}]\,.
\end{cases}
$$

To know if the condition of formal symmetry is also *sufficient* we must decide what conditions must satisfy δu and its derivatives on the boundary. Precisely, if p is the maximum order of the derivatives appearing in the formal operator $\mathfrak{S}$, the formal symmetry becomes also *sufficient* when considering δu and its derivatives up to the order $p - 1$ vanishing on the boundary.

These requirements on the variation δu and on its derivatives can be weakened in the following cases:

1) if the formal operator is of even order, say $2p$, or of odd order, say $2p+1$, the condition of formal symmetry is *sufficient* provided that δu and its derivatives up to the order p vanish on the boundary;

2) if with the formal operator are associated boundary conditions that make the operator *symmetric* then the sole condition that u belongs to the domain of the operator is sufficient to assure the existence of the variational formulation [1]; that is, we need not impose supplementary requirements on δu at the boundary. In particular the same condition is valid if the operator is *self-adjoint* that is if boundary conditions for the operator S are the same than those for the operator $\tilde{S}$, so $D(S) = D(\tilde{S})$;

3) when *natural* boundary conditions are associated with a formal operator, on a piece of the boundary, then δu need not satisfy conditions on the piece for deducing the equation from a functional.

Hamilton's principle belongs to case 1); Dirichlet's problem for the potential equation belongs to case 2); Neumann's problem for the potential equation belongs to case 3).

We show here that the condition of formal symmetry is sufficient if δu and its derivatives, up to the order $p-1$, vanish on the boundary. The less pretentious cases 1), 2), 3) can be easily deduced from this case: see [1], if $\mathfrak{S}$ is formally symmetric of order p, multiplying the equation (8) by δu and integrating we obtain:

$$\text{(21)} \qquad 0 = \int_{\Omega} \delta u\,(\mathfrak{S}u)\,d\Omega = \delta \int_{\Omega} \frac{1}{2}\,u\,\mathfrak{S}u\,d\Omega + \{\text{boundary terms}\}.$$

But boundary terms contain linearly δu and its derivatives up to the order $p-1$, which vanish on the boundary by hypothesis. Then the equation (8) comes from the functional:

$$\text{(22)} \qquad \mathfrak{J}[u] = \int_{\Omega} \frac{1}{2}\,u\,\mathfrak{S}u\,d\Omega$$

with the condition $\delta\mathfrak{J}[u] = 0$ and δu and its derivatives up to the order $p-1$ vanish on the boundary. With these conditions the formal symmetry becomes sufficient.

4. Variational formulation for two systems of differential equations.

We pose the following question: what are the conditions for two systems of differential equations to be deduced from the stationarity of the same functional? The following theorem holds:

THEOREM II.—The necessary condition for deducing two systems of linear differential equations:

$$\text{(23)} \qquad \mathfrak{S}u = 0 \qquad \mathfrak{T}v = 0$$

from the stationarity of a single functional with respect to arbitrary variations of the two tensors u and v is that the two formal operators $\mathcal{S}$ and $\mathcal{T}$ be one the formal adjoint of the other:

$$\mathcal{T} = \widetilde{\mathcal{S}}. \tag{24}$$

Proof: If two systems of equations follow from a single functional the lagrangian must be linear in u and v. The most general bilinear lagrangian is:

$$L(u, v) = \Sigma_s w_s (\mathcal{R}_s u)(\mathcal{T}_s v) \tag{25}$$

where $\mathcal{R}_s$ and $\mathcal{T}_s$ are two formal operators and w_s tensors. Now

$$\delta\mathfrak{I}[u, v] = \int_\Omega \Sigma_s [w_s (\mathcal{R}_s \delta u)(\mathcal{T}_s v) + w_s (\mathcal{R}_s u)(\mathcal{T}_s \delta v] \, d\Omega = \tag{26}$$

$$= \int_\Omega [\delta u (\Sigma_s \widetilde{\mathcal{R}}_s w_s \mathcal{T}_s v) + \delta v (\Sigma_s \widetilde{\mathcal{T}}_s w_s \mathcal{R}_s u)] \, d\Omega + \{\text{boundary terms}\}.$$

Then if we put $\delta\mathfrak{I} = 0$ for the arbitrariness of δu and δv it follows that:

$$[\Sigma_s \widetilde{\mathcal{T}}_s w_s \mathcal{R}_s] u = 0 \qquad [\Sigma_s \widetilde{\mathcal{R}}_s w_s \mathcal{T}_s] v = 0. \tag{27}$$

The formal adjoint of the first operator is just the second one, as can be seen with an integration by parts.

The condition (24) becomes also sufficient if δu and δv vanish with their derivatives up to a certain order on the boundary. In fact multiplying the first of (23) by δv, the second by δu, integrating on Ω and adding we obtain:

$$\int_\Omega (\delta v \, \mathcal{S} u + \delta u \, \widetilde{\mathcal{S}} v) \, \delta\Omega = 0. \tag{28}$$

Being $\delta v = \delta u = 0$ on the boundary with its derivatives up to the order that appears in the boundary integrals, the equation (28) becomes:

$$\delta \int_\Omega v \, \mathcal{S} u \, d\Omega = 0. \tag{29}$$

As an example, let us consider the two systems:

$$\operatorname{div} \vec{u} = \rho \qquad -\operatorname{grad} v = \vec{s}. \tag{30}$$

Because div and —grad are formally adjoint we have the variational formulation

$$\delta \int_\Omega (v \operatorname{div} \vec{u} - v\rho - \vec{u} \cdot \vec{s}) \, d\Omega = 0 \tag{31'}$$

from which the two systems (30) follow, assuming $\delta u = 0$ on the boundary.

Thus we have shown that the variational formulation of one or more systems of differential equations only requires a particular structure of the *formal* differential operator *whatever the boundary conditions may be*. To investigate the minimum or the maximum of the functional we must consider also the boundary conditions [3].

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