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An analog method for computing the constrained minimum of a convex quadratic function

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Analisi numerica. — *An analog method for computing the constrained minimum of a convex quadratic function.* Nota di GIUSEPPE BASILE e GIOVANNI MARRO, presentata (*) dal Corrisp. G. EVANGELISTI.

RIASSUNTO. — Argomento del presente lavoro è una possibile soluzione mediante calcolatore analogico del problema della ricerca del minimo di una funzione convessa di più variabili in presenza di vincoli di tipo saturazione.

Viene descritto dapprima un semplice modello in retroazione per la realizzazione delle condizioni necessarie e sufficienti di Kuhn e Tucker.

La stabilità del modello proposto viene quindi analizzata e provata applicando il metodo diretto di Liapounoff.

Viene infine presentata una possibile applicazione del procedimento alla soluzione mediante calcolatore ibrido di un particolare problema di ottimizzazione dinamica.

1. INTRODUCTION.

By means of Pontryagin's maximum principle or other variational methods many dynamic optimization problems are reduced to two-point boundary value problems, which usually are solved by iterative procedures [1–5].

A mechanization of these procedures can be obtained both by means of digital and hybrid computers: when a digital computer is used, the computation of the minimum of the hamiltonian function is performed at each step of the integration by the well known techniques of nonlinear programming or by especially developed iterative procedures [6]; on the other hand, when the problem is solved by an hybrid computer, it is necessary to minimize the halmiltonian at every instant of time by means of a proper analog circuit, which is requested to be sufficiently prompt in response and versatile enough to allow for time dependency of some coefficients of the hamiltonian.

The aim of the present paper is to study a possible realization of an analog circuit for the instantaneous search of the minimum of a quadratic convex function (i.e. the hamiltonian in minimum-energy problems) in presence of saturation bounds on the variables. In this case the well known Kuhn and Tucker theorem gives necessary and sufficient conditions for the minimum expressed by a set of algebraic equations, which must be mechanized in order to solve the problem: being equations in implicit form, feedback is necessary and then a stability problem arises.

As in general it is done for overcoming stability difficulties, it is necessary to solve on the analog computer a set of differential equations whose equilibrium points are the solutions of the given set of algebraic equations and whose stability is a priori ensured [7–9].

(*) Nella seduta del 13 gennaio 1968.

In the particular case under consideration, where the algebraic equations are nonlinear, the stability of the corresponding differential model is investigated by means of the Liapounov direct method and it is shown that it is always possible to obtain a stable system by a proper choice of a set of free parameters.

2. Necessary and sufficient conditions for the minimum and their analog model.

Let

$$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}\cdot\mathbf{A}\boldsymbol{x} + \boldsymbol{b}\cdot\boldsymbol{x} \tag{2.1}$$

be a quadratic function given in R^n; the $n \times n$ matrix $\mathbf{A}$ is assumed to be symmetric and positive definite, while the n–vector $\boldsymbol{b}$ is arbitrary. It is required to find the value $\boldsymbol{x}^*$ of $\boldsymbol{x}$ which minimizes $f(\boldsymbol{x})$ over the hypercubic constraint set

$$\mathfrak{X} = \{\boldsymbol{x} : |x_i| \leq 1 \ (i = 1, 2, \cdots, n)\} \tag{2.2}$$ (1).

Being function (2.1) convex and set (2.2) convex and compact, in the minimum point $\boldsymbol{x}^*$ the necessary and sufficient condition

$$-(\mathbf{grad} f)_{\boldsymbol{x}^*} \in \mathcal{C}'(\mathfrak{X} - \boldsymbol{x}^*) \tag{2.3}$$

holds, where $\mathcal{C}'(\mathfrak{X} - \boldsymbol{x}^*)$ denotes the dual cone of the set $\mathfrak{X}$ at the point $\boldsymbol{x}^*$, defined as

$$\mathcal{C}'(\mathfrak{X} - \boldsymbol{x}^*) = \{\boldsymbol{p} : \boldsymbol{p}\cdot(\boldsymbol{x} - \boldsymbol{x}^*) \leq 0 \quad \forall \boldsymbol{x} \in \mathfrak{X}\} \tag{2.4}$$ (2).

For the particular problem here considered, from equations (2.3) and (2.4) can be deduced the simpler necessary and sufficient condition

$$-\mathbf{A}\boldsymbol{x}^* - \boldsymbol{b} = \boldsymbol{p}, \tag{2.5}$$

where $\boldsymbol{p}$ is an n–vector such that

$$\begin{cases} p_i \leq 0 & \text{if } x_i^* = -1, \\ p_i = 0 & \text{if } |x_i^*| < 1, \\ p_i \geq 0 & \text{if } x_i^* = 1 \end{cases} \qquad (i = 1, 2, \cdots, n). \tag{2.6}$$

(1) The more general case when the constraint set is an hyperparallelepiped

$$\mathfrak{X} = \{\boldsymbol{x} : X_{i\,\min} \leq x_i \leq X_{i\,\max} \ (i = 1, 2, \cdots, n)\}$$

can be easily reduced to the above stated problem using a simple linear transformation.

(2) Condition (2.3) is substantially equivalent to Kuhn-Tucker condition for the minimum of a convex function with convex inequality constraints.

A proof of (2.3) can be found in [10].

Remark that a solution of the stated minimization problem exists because the set $\mathfrak{X}$ is compact and it is unique because the function $f(\boldsymbol{x})$ is strictly convex.

A possible analog computer implementation for the automatic solution of the problem is shown in fig. 1. Δ indicates an arbitrary positive definite diagonal $n \times n$ matrix, which can be properly chosen in order to assure the stability of the feedback system, as will be shown in the next section.

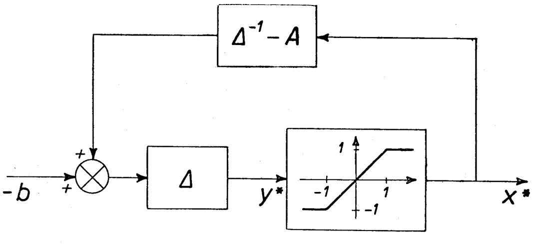

Fig. 1. – An algebraic feedback model for the minimization problem.

Note that the input vector $\boldsymbol{b}$ can be time dependent and therefore the proposed circuit is particularly useful in dynamic optimization problems, where it is required to maximize a time dependent hamiltonian function (see section 4).

The nonlinear block which appears in fig. 1 corresponds to a set of saturations. If by

$$\boldsymbol{x} = \mathbf{sat}\,(\boldsymbol{y}) \tag{2.7}$$

it is meant that

$$\begin{cases} x_i = -1 & \text{if } \ y_i \leq -1, \\ x_i = y_i & \text{if } \ |y_i| < 1, \\ x_i = 1 & \text{if } \ y_i \geq 1 \end{cases} \qquad (i = 1, 2, \cdots, n), \tag{2.8}$$

an equilibrium point of the system is defined by

$$\boldsymbol{x}^* = \mathbf{sat}\,((\mathbf{I} - \Delta\mathbf{A})\,\boldsymbol{x}^* - \Delta\boldsymbol{b}), \tag{2.9}$$

where $\mathbf{I}$ is the unit matrix of order n.

The equivalence of (2.9) and (2.5), (2.6) is easily proved remarking that, if $\boldsymbol{x}^* = \mathbf{sat}\,(\boldsymbol{y}^*)$, it is true that $\boldsymbol{y}^* - \boldsymbol{x}^* = \boldsymbol{p}'$, where $\boldsymbol{p}'$ is an n-vector which satisfies conditions (2.6); therefore (2.9) leads to

$$-\mathbf{A}\boldsymbol{x}^* - \boldsymbol{b} = \Delta^{-1}\boldsymbol{p}', \tag{2.10}$$

which is just the same as (2.5) because, if $\boldsymbol{p}'$ satisfies (2.6), also $\Delta^{-1}\boldsymbol{p}'$ does.

From the uniqueness of the solution of the stated minimization problem and sufficiency of (2.5) and (2.6) it follows that the analog model has an unique equilibrium point.

Being the analog implementation shown in fig. 1 a feedback system, a stability investigation is necessary in order to be sure that the above mentioned equilibrium point will be actually reached.

3. The stability of the model.

In order to investigate the stability, it is necessary to make assumptions on the dynamics of the system.

In light of the uncertainty in the dynamic behaviour of algebraic computing units, it is convenient to place somewhere in the loop a diagonal matrix of properly dimensioned lags.

Assuming equal values for all the lags, their position in the loop becomes immaterial, so that they can be placed as it is shown in fig. 2, where the added block represents a diagonal matrix of transfer functions, which is supposed to be the sole responsible for system dynamics.

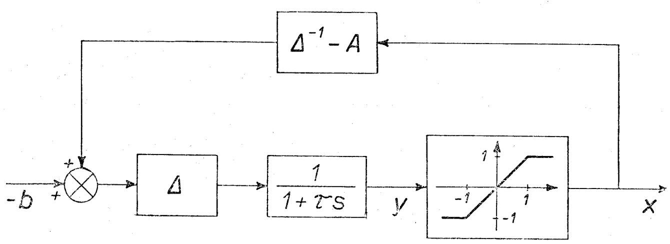

Fig. 2. – The feedback model modified in order to take into account the dynamic behaviour.

The evolution of the system is then described by the vector differential equation

$$\tau \dot{\boldsymbol{y}} = -\boldsymbol{y} + (\mathbf{I} - \Delta \mathbf{A}) \,\mathbf{sat}\,(\boldsymbol{y}) - \Delta \boldsymbol{b}, \tag{3.1}$$

where $\boldsymbol{y}$ is the input vector of the nonlinear block.

The equilibrium point $\boldsymbol{y}^*$ is obviously the solution of the steady-state equation corresponding to (3.1):

$$\mathbf{o} = -\boldsymbol{y}^* + (\mathbf{I} - \Delta \mathbf{A}) \,\mathbf{sat}\,(\boldsymbol{y}^*) - \Delta \boldsymbol{b}. \tag{3.2}$$

Let

$$\boldsymbol{u} = \boldsymbol{y} - \boldsymbol{y}^* \tag{3.3}$$

denote the variation about the equilibrium point, which satisfies the differential equation

$$\tau \dot{\boldsymbol{u}} = -\boldsymbol{u} + (\mathbf{I} - \Delta \mathbf{A}) (\mathbf{sat}\,(\boldsymbol{u} + \boldsymbol{y}^*) - \mathbf{sat}\,(\boldsymbol{y}^*)). \tag{3.4}$$

obtained by subtracting (3.2) from (3.1), every solution of which is required to tend to the origin as time increases indefinitely, in order for the equilibrium point $\boldsymbol{y}^*$ to be asymptotically stable.

A component of the vector function **sat** $(\boldsymbol{y})$, i.e. the scalar function sat (y), is plotted in fig 3, *a*: it is easy to verify that the graph of sat $(u + y^*)$ — — sat (y^*), which is a scalar function of u, can be obtained from the graph of sat (y) by moving the origin of axes into the point $(y^*$, sat $(y^*))$, as is shown in fig. 3, *b*. The same holds, of course, also for the corresponding vector function, which appears in the right side member of equation (3.4).

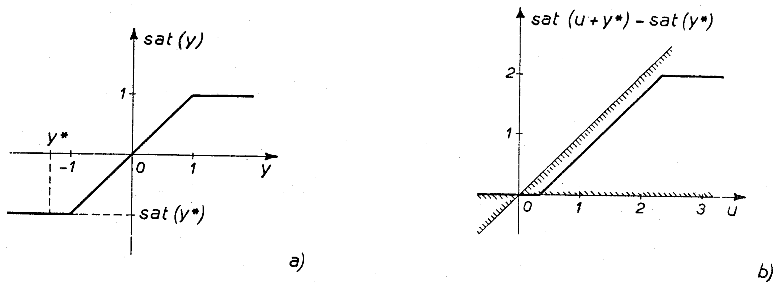

Fig. 3 *a*, *b*. – Some particular properties of the function $x =$ sat (y).

In view of applying Liapounouv's direct method it is convenient to re-write equation (3.4) in the form

(3.5) $$\tau\dot{\boldsymbol{u}} = -\boldsymbol{u} + (\mathbf{I} - \Delta\mathbf{A})\,\mathbf{G}\boldsymbol{u}\,,$$

where $\mathbf{G}$ is a diagonal matrix whose elements are functions of $\boldsymbol{u}$ and $\boldsymbol{y}^*$ not less than zero and not greater than one for every couple of values of $\boldsymbol{u}$ and $\boldsymbol{y}^*$. This condition requires that every component of the nonlinear function in the right side member of equation (3.4) lies in the shaded sector shown in fig. 3, *b* and it is clearly satisfied in the problem considered, but it is more general: in fact it corresponds to an Aizerman absolute stability problem for many nonlinearities.

Let us assume the Liapounouv function

(3.6) $$\mathrm{V}(\boldsymbol{u}) = \frac{1}{2}\,\boldsymbol{u}\cdot\Delta^{-1}\boldsymbol{u}.$$

whose time derivative along a trajectory of equation (3.5) is

(3.7) $$\dot{\mathrm{V}}(\boldsymbol{u}) = \tau^{-1}(\Delta^{-1}\boldsymbol{u})\cdot(-\boldsymbol{u} + (\mathbf{I} - \Delta\mathbf{A})\,\mathbf{G}\boldsymbol{u}) =$$
$$= \tau^{-1}(-\boldsymbol{u}\cdot\Delta^{-1}\boldsymbol{u} + \boldsymbol{u}\cdot(\Delta^{-1} - \mathbf{A})\,\mathbf{G}\boldsymbol{u}).$$

Now to prove the stability it will be shown that such a quadratic form is negative definite for a proper choice of Δ and for every $\mathbf{G}$ satisfying the above mentioned conditions.

In fact, by the nonsingular transformation

(3.8) $$\boldsymbol{u} = \sqrt{\Delta}\,\boldsymbol{z},$$

quadratic form (3.7) becomes

$$(3.9) \qquad \tau^{1-}(-\boldsymbol{z}\cdot\boldsymbol{z} + \boldsymbol{z}\cdot(\mathbf{I} - \sqrt{\Delta}\mathbf{A}\sqrt{\Delta})\,\mathbf{G}\boldsymbol{z}),$$

which clearly has the same sign definition as (3.7).

Neglecting the positive factor τ^{-1}, (3.9) can be written

$$(3.10) \qquad -\boldsymbol{z}\cdot\boldsymbol{z} + \boldsymbol{z}\cdot\mathbf{B}\boldsymbol{z},$$

where $\mathbf{B} = (\mathbf{I} - \sqrt{\Delta}\mathbf{A}\sqrt{\Delta})\,\mathbf{G}$.

Being

$$(3.11) \qquad \boldsymbol{z}\cdot\boldsymbol{z} = \|\boldsymbol{z}\|^2,$$

$$(3.12) \qquad |\boldsymbol{z}\cdot\mathbf{B}\boldsymbol{z}| \leq \|\boldsymbol{z}\|\,\|\mathbf{B}\boldsymbol{z}\| \leq \|\mathbf{B}\|\,\|\boldsymbol{z}\|^2,$$

quadratic form (3.10) is certainly negative definite if $\|\mathbf{B}\| < 1$; on the other hand

$$(3.13) \qquad \|\mathbf{B}\| \leq \|\mathbf{I} - \sqrt{\Delta}\mathbf{A}\sqrt{\Delta}\|\,\|\mathbf{G}\|$$

and, since $\|\mathbf{G}\| < 1$, it is sufficient to prove that, by a proper choice of Δ, also $\|\mathbf{I} - \sqrt{\Delta}\mathbf{A}\sqrt{\Delta}\|$ can be made less than 1.

As is well known, the norm of a linear transformation expressed by a matrix $\mathbf{Q}$ is the square root of the largest eigenvalue of the matrix $\mathbf{Q}^T\mathbf{Q}$: it is obvious that the norm of a symmetric matrix is the largest absolute value of its eigenvalues.

Let $\lambda_i\,(i = 1, 2, \cdots, n)$ be the eigenvalues of $\sqrt{\Delta}\mathbf{A}\sqrt{\Delta}$, which are all real and positive, being $\mathbf{A}$ symmetric and positive definite; then the eigenvalue of $\mathbf{I} - \sqrt{\Delta}\mathbf{A}\sqrt{\Delta}$ are $\mu_i = 1 - \lambda_i\,(i = 1, 2, \cdots, n)$, so that if $\lambda_i < 2$, all μ_i satisfy the inequality $|\mu_i| < 1$ and therefore $\|\mathbf{I} - \sqrt{\Delta}\mathbf{A}\sqrt{\Delta}\| < 1$.

Whatever the positive definite matrix $\mathbf{A}$ is, it is possible to make the elements on the main diagonal of $\sqrt{\Delta}\mathbf{A}\sqrt{\Delta}$ equal to arbitrary real positive numbers by a proper choice of the elements of Δ. On the other hand it is well known that the sum of the eigenvalues of an arbitrary square matrix is equal to the trace of the matrix, i.e. the sum of the elements on the main diagonal. Therefore, in order to secure the stability of the feedback analog model, it is sufficient to choose the positive definite diagonal matrix Δ in such a way that the trace of $\sqrt{\Delta}\mathbf{A}\sqrt{\Delta}$ is less than 2.

4. Maximization of an hamiltonian function by the outlined method.

Let consider the problem of fixed-time optimal control of a linear time-invariant system described by the vector differential equation

$$(4.1) \qquad \dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u},$$

where $\boldsymbol{x} \in \mathrm{R}^n$ and $\boldsymbol{u} \in \mathrm{R}^m$ denote the state and control vectors, from a given initial state $\boldsymbol{x}_a$ to a given admissible terminal state $\boldsymbol{x}_b$, in presence of bounds on controls expressed by the inequalities

$$|u_i| \leq 1 \qquad (i = 1, 2, \cdots, m) \tag{4.2}$$

and the performance index

$$J = \frac{1}{2} \int_{t_a}^{t_b} (\boldsymbol{x} \cdot \mathbf{Q} \boldsymbol{x} + \boldsymbol{u} \cdot \mathbf{R} \boldsymbol{u} + 2\, \boldsymbol{x} \cdot \mathbf{M} \boldsymbol{u})\, dt\,, \tag{4.3}$$

consisting in the integral of a positive definite quadratic function of the state and control.

This optimization problem is well known and has been treated for instance by Athans and Falb [11]; a direct analytic solution is possible when the control variables are not bounded (see Kalman [12]), but in the more general case when there are bounds on the controls the stated problem, which reduces to a nonlinear two-point boundary value problem, must be solved by iterative procedures.

By virtue of Pontryagin's necessary conditions, which in the case here considered are also sufficient if the terminal state $\boldsymbol{x}_b$ is an internal point of the reachable set (see Lee, Mangasarian and the autors [13–15]), if $\boldsymbol{x}^*(t)$ is an optimal trajectory, corresponding to the control $\boldsymbol{u}^*(t)$, there exists a vector $\boldsymbol{p}_a$ such that the solution of the adjoint system

$$\boldsymbol{p} = -\mathbf{A}^{\mathrm{T}} \boldsymbol{p} + \mathbf{Q} \boldsymbol{x}^* + \mathbf{M} \boldsymbol{u}^* \tag{4.4}$$

with initial condition

$$\boldsymbol{p}(t_a) = \boldsymbol{p}_a \tag{4.5}$$

satisfies the maximum condition

$$\mathrm{H}(\boldsymbol{x}^*, \boldsymbol{u}^*, \boldsymbol{p}) \geq \mathrm{H}(\boldsymbol{x}^*, \boldsymbol{u}, \boldsymbol{p}) \tag{4.6}$$

for every admissible $\boldsymbol{u}$ and at every instant of time; the hamiltonian function H is given by

$$\mathrm{H}(\boldsymbol{x}^*, \boldsymbol{u}, \boldsymbol{p}) = -\frac{1}{2} (\boldsymbol{x}^* \cdot \mathbf{Q} \boldsymbol{x}^* + \boldsymbol{u} \cdot \mathbf{R} \boldsymbol{u} + 2\, \boldsymbol{x}^* \cdot \mathbf{M} \boldsymbol{u}) + \boldsymbol{p} \cdot (\mathbf{A} \boldsymbol{x}^* + \mathbf{B} \boldsymbol{u})\,. \tag{4.7}$$

Inequality (4.6) implies that at every instant of time the convex quadratic function $-\mathrm{H}$ is minimized by $\boldsymbol{u}^*$ over the set (4.2). According to equation (2.9), in this particular case the maximum condition is equivalent to the relationship

$$\boldsymbol{u}^* = \mathbf{sat}\, ((\mathbf{I} - \Delta \mathbf{R})\, \boldsymbol{u}^* + \Delta\, (\mathbf{B}^{\mathrm{T}} \boldsymbol{p} - \mathbf{M}^{\mathrm{T}} \boldsymbol{x}^*))\,, \tag{4.8}$$

where Δ is an arbitrary positive definite diagonal matrix (3).

(3) For this problem Athans and Falb [11, page 479] gave the explicit solving formula

$$\boldsymbol{u}^* = \mathbf{sat}\, (\mathbf{R}^{-1} (\mathbf{B}^{\mathrm{T}} \boldsymbol{p} - \mathbf{M}^{\mathrm{T}} \boldsymbol{x}^*)),$$

which is incorrect in the general case when $\mathbf{R}$ is not a diagonal matrix.

Equation (4.8) defines implicitly the value of the optimal control at every instant of time and its solution, as it has been previously remarked, exists and is unique; therefore every trajectory $\boldsymbol{x}^*(t)$, $\boldsymbol{u}^*(t)$ which satisfies equations (4.1), (4.4) and (4.8) is an optimal trajectory, and the optimal control problem can be solved by means of the analog model shown in fig. 4, where

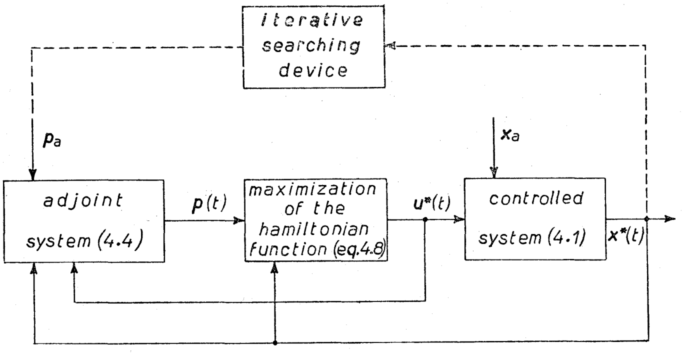

Fig. 4. – Device for the computation of the optimal control law.

the maximization block is implemented as in fig. 2, and the initial value of the adjoint vector $\boldsymbol{p}_a$ is adjusted by a trial and error procedure or a gradient method in order to meet the desired terminal state $\boldsymbol{x}_b$.

5. Conclusion.

It has been shown that the problem of minimizing a convex function given by the sum of a positive definite quadratic form and linear form with independent bounds on every variable can be solved by means of a simple nonlinear analog feedback model which at the equilibrium state satisfies the Kuhn-Tucker necessary and sufficient conditions and can be always made asymptotically stable by a proper choice of a set of free parameters.

The method is particularly useful for solving a class of dynamic optimization problems on analog or hybrid computers because it allows for a continuous search of the maximum of hamiltonian functions where the coefficients of the linear terms are time dependent.

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