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Quasi-Steiner Systems

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Matematica. — *Quasi-Steiner Systems.* Nota di JONATHAN DEAN SWIFT (*), presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Si risolvono i problemi di trovare il numero massimo (risp. minimo) di terne su n lettere, tali che nessuna coppia compaia in più di una terna (risp. ogni coppia compaia in almeno una terna). Inoltre si stabilisce un limite per un problema più generale, proposto da B. Segre.

1. INTRODUCTION.

The problem of the existence of sets of triples on n letters such that every pair of letters appears in one and only one triple was raised by Steiner [5] more than a century ago. It now appears [2] that the problem had been resolved by Kirkman [3] some years before Steiner posed it but the sets of triples meeting this condition are called Steiner Triple Systems. A number of constructions of these systems for the possible values of n are known, see, e.g. [1] and [2].

In 1958, Fort and Hedlund [1] solved a generalization of the Steiner problem which we indicate in this paper as problem 1:

To find the minimum number of triples on n letters such that every pair of letters appears in at least one triple.

In his Lectures on Higher Geometry, Segre has proposed a general problem with applications to the existence of certain types of nets [4], p. 245, which, for the case $t = 4$ of the problem as stated, reduces to our problem 2:

To find the maximum number of triples on n letters such that no pair of letters appears in more than one triple.

The purposes of this note are to indicate the solution to problem 2 and its very close connection with problem 1. We cannot speak an of exact duality since problem 2 requires a parity argument which problem 1 does not, but the essential equivalence of the solutions, this one argument apart, will be apparent in the sequel. For this reason, we solve both problems in parallel. The solution of problem 1 simplifies and, perhaps, clarifies that given in (1). We shall use the results of (1) directly only for the case $n = 6k + 5$.

2. THE BOUNDS. SPECIAL CASE OF PROBLEM 2 FOR $n = 6k + 5$.

Problem 2: No letter may occur in more than $\left[\frac{n-1}{2}\right]$ triples (where $[x]$ is the greatest integer $\leq x$) since it must always appear with

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disjoint pairs of other letters. Since there are n letters, three pairs to a triple, there cannot be more than $\left[\frac{n\left[\frac{n-1}{2}\right]}{3}\right]$ triples.

P r o b l e m 1 : Every letter must appear with the $n-1$ other letters. This will require each letter to appear in at least $\left\{\frac{n-1}{2}\right\}$ triples (where $\{x\}$ is the least integer $\geq x$, $\{x\} = -[-x]$). Thus there must be at least $\left\{\frac{n\left\{\frac{n-1}{2}\right\}}{3}\right\}$ triples.

If $n \equiv 5$ (mod. 6) the limit above would give $\frac{n(n-1)-2}{6}$ triples for problem 2. Each triple includes three pairs so that $\frac{n(n-1)}{2} - 1$ pairs would be included if that limit is reached, or all pairs except one. This situation cannot happen. Perhaps the simplest way to see this is to use ordered pairs. A system of triples, maximal or not, has an incidence matrix, an $n \times n$ matrix in which the element in the ith row and jth column is 1 if the ordered pair (i, j), $i \neq j$, occurs in a triple of the system and zero otherwise. A third symbol, say -1, may be used to fill in the diagonal entries. Any triple produces 6 entries of 1, two in each of three rows and each of three columns. In the conditions of problem 2 no overlap of these entries can occur. Therefore, each row and each column has an even number of entries of zero and of one. If a single pair (a, b) remained uncovered, there would be a single zero in the positions (a, b) and (b, a) of the matrix and the ath and bth rows would have an odd number (1) of zero entries.

Thus, when $n \equiv 5$ (mod 6), the largest possible number of triples for problem 2 is one less than the previously calculated limit or $\frac{n^2-n-8}{6}$. We list the values of the limits for each problem and each residue class modulo 6. For problem 1, the bounds are lower, for problem 2 they are upper.

n (mod 6)	*Problem 1*	*Problem 2*
0	$\frac{n^2}{6}$	$\frac{n^2-2n}{6}$
1	$\frac{n^2-n}{6}$	$\frac{n^2-n}{6}$
2	$\frac{n^2+2}{6}$	$\frac{n^2-2n}{6}$
3	$\frac{n^2-n}{6}$	$\frac{n^2-n}{6}$
4	$\frac{n^2+2}{6}$	$\frac{n^2-2n-2}{6}$
5	$\frac{n^2-n+4}{6}$	$\frac{n^2-n-8}{6}$.

3. The main result.

THEOREM 1. *The bounds as defined above can always be attained for* $n \geq 3$.

Proof. For $n \equiv 1, 3 \pmod 6$ the bounds of both problems are identical and both problems are solved by the known Steiner Triple Systems for these orders. We shall not repeat here a construction of such systems.

For $n \equiv 5 \pmod 6$ we use the constructions of Fort and Hedlund [1] for this case. These constructions, aimed at problem 1, produce $(n^2 - n + 4)/6$ triples containing all pairs but one, say (a, b), once and (a, b) three times. To solve problem 2, we simply reject two of the three triples containing (a, b) and have $(n^2 - n + 4)/6 - 2 = (n^2 - n - 8)/6$ triples satisfying the requirements of problem 2.

For $n \equiv 0, 2 \pmod 6$ of problem 2, we delete from a Steiner Triple System of order $n + 1$ all triples containing some one letter which we may designate by a. Now a appears with n letters or in $n/2$ triples. From $(n + 1)\, n/6$ triples we have eliminated $n/2$, leaving $\frac{n^2 + n - 3n}{6} = \frac{n^2 - 2n}{6}$.

For $n \equiv 2, 4 \pmod 6$ of problem 1, we use the Steiner Triple System on $n - 1$ letters, adjoin an nth letter, say n, and form triples with n using pairs of the original $n - 1$ letters as neatly as we can. (One overlap of the type (n, a, b), (n, a, c) *must* occur since $n - 1$ is odd.) There result $(n - 1)(n - 2)/6 + n/2 = (n^2 + 2)/6$ triples containing every pair at least once (and $n/2 + 1$ twice).

For $n \equiv 4 \pmod 6$ of problem 2, let (a, b) be the pair occurring three times in the Fort-Hedlund construction for $n + 1$. Then, when two of the triples containing (a, b) are eliminated to solve problem 2, a appears with only $n - 2$ other elements, or in $(n - 2)/2$ triples. Delete these triples. We have

$$\frac{(n+1)^2 - (n+1) - 8}{6} - \frac{(n-2)}{2} = \frac{n^2 - 2n - 2}{6}$$

triples on n letters satisfying problem 2.

For $n \equiv 0 \pmod 6$ of problem 1, use once more the Fort-Hedlund construction, this time for $n - 1$ letters. Adjoin an nth letter which we designate n. Let the triples which contained the pair appearing three time be (a, b, c), (a, b, d), (a, b, e).

Replace these by (a, b, c), (n, b, d), (n, b, e), (a, d, e). All pairs appearing before still appear and there are three pairs including n. Divide the $n - 1 - 3 = n - 4$ letters which have not appeared with n into $(n - 4)/2$ disjoint pairs and complete them to triples with n. There are

$$\frac{(n-1)^2 - (n-1) + 4}{6} - 3 + 4 + \frac{(n-4)}{2} = \frac{n^2}{6}$$

triples and all pairs are included.

4. CANONICITY OF THE CONSTRUCTIONS.

In [1] it is proved that in *any* minimal construction for problem 1 the pattern of repeated pairs is the same. We have already mentioned this pattern for $n \equiv 5 \pmod 6$, one pair repeated three times, and for $n \equiv 2,4 \pmod 6$, $n/2 + 1$ pairs appear twice. For $n \equiv 0 \pmod 6$ an inspection will show that $n/2$ pairs appear twice (the $(n-4)/2$ last adjoined and (b, n) and (d, e)). For $n \equiv 1, 3 \pmod 6$ there are, of course, no repeated pairs.

A similar situation applies to problem 2.

THEOREM 2. *In any system of triples solving problem* 2, *there are* $n/2$ *omitted pairs, all disjoint, if* $n \equiv 0, 2 \pmod 6$; $n/2 + 1$ *omitted pairs such that* $n - 1$ *letters appear in only one omitted pair and one letter appears in three pairs for* $n \equiv 4 \pmod 6$; *and* 4 *omitted pairs involving* 4 *letters symmetrically if* $n \equiv 5 \pmod 6$.

Proof. That the total number of omitted pairs has the value given can be calculated simply by subtracting three times the bound given (each triple has three pairs) from the total number of pairs, $n(n-1)/2$.

For $n \equiv 0,2 \pmod 6$, each letter may appear with at most $n - 2$ others. Thus each letter must appear in at least one omitted pair. If one appeared in more than one, the bound could not be attained.

For $n \equiv 4 \pmod 6$, again each element must appear in an omitted pair. A parity argument similar to that given for $6k + 5$ in § 2 shows that each element must appear in an odd number of omitted pairs. Since the are only $(n + 2)/2$ omitted pairs, one element and only one can appear in three of these pairs.

For $n \equiv 5 \pmod 6$ parity would permit an omission of the type (a, b), (a, c), (a, d), (a, e). If this happened, a would appear with only $n - 5$ letters. If we omit the $(n-5)/2$ triples containing a, we would have a solution for $n - 1$ containing $(n^2 - n - 8)/6 - (n-5)/2 = (n^2 - 4n + 7)/6 = \frac{(n-1)^2 - 2(n-1) + 4}{6}$ triples or one more than the maximum. (We could also argue that the addition of triples (a, b, c), (a, d, e) would yield a noncanonical solution for problem 1).

Remark. It follows from a comparison of these totals that, for $n \equiv 0 \pmod 2$, $n > 4$, *no* solution of problem 1 can be obtained by adding triples to a solution of problem 2 and viceversa. The contrary situation applies to $n \equiv 5 \pmod 6$ where a solution to one problem may *always* be obtained from a solution to the other by adding or deleting triples as the case may be.

5. BOUNDS FOR THE GENERAL PROBLEM.

In this concluding section we shall discuss briefly the general problem posed by Segre [4]. The remarks, in a slightly simplified form may be translated directly to the corresponding generalization of problem 1.

P r o b l e m 2′. *To find the maximum number of k–tuples on n letters such that no (k — 1)–uple appears in more than one k–tuple.*

If we have any set of k–tuples satisfying the conditions of problem 2′ for n, and if we select all the k–tuples containing some designated mark, say a, and delete a from these k–tuples, the result will be a set of $(k-1)$–uples satisfying the conditions of problem 2′ for $n-1$. Thus, denoting the answer to problem 2′ by $f(n, k)$, we have $f(n, k) \leq \left[\frac{nf(n-1, k-1)}{k}\right]$.

In an imperfect analogy with $\binom{n}{k}$, define $\left[\begin{matrix} n \\ k \end{matrix}\right]$ by $\left[\begin{matrix} n \\ k \end{matrix}\right] = \left[\frac{n\left[\begin{matrix} n-1 \\ k-1 \end{matrix}\right]}{k}\right]$ for $k \geq 2$, $n \geq k$, $\left[\begin{matrix} n \\ 1 \end{matrix}\right] = 1$. Then $f(n, k) \leq \left[\begin{matrix} n \\ k \end{matrix}\right]$.

It may be conjectured that $\left[\begin{matrix} n \\ k \end{matrix}\right]$ is, in some sense, a «good» bound. Perhaps $f(n, k) = \left[\begin{matrix} n \\ k \end{matrix}\right] - o(n^c)$ where c may depend on k but not on n, $c < k$. The proportion of cases where $f(n, k) = \left[\begin{matrix} n \\ k \end{matrix}\right]$ inevitably decreases as k increases. For example, when $n = 9$, $k = 5$, $\left[\begin{matrix} n \\ k \end{matrix}\right] = 25$ which would imply only one omitted quadruple. This situation cannot exists for the same reason we could not have a single omitted pair. However, for $n = 8$, $k = 4$, there is a «perfect» solution where each triple occurs exactly once. Nevertheless, it is entirely reasonable to suppose that $\left[\begin{matrix} n \\ k \end{matrix}\right]$ is always attainable for some n for each k. Similar conjectures occur for designs [2].

ADDED IN PROOF: A number of the results stated also appear (in a different formulation) in a paper by J. SCHÖNHEIM, *On Maximal Systems of k-tuples*, «Studia Scientiarum Math. Hung.», *1*, 363–368 (1966). The author thanks prof. Hedlund for calling his attention to this work.

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