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Periodic or almost-periodic solutions of a non linear functional equation. Nota IV

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 13 gennaio 1968

Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi matematica. — *Periodic or almost-periodic solutions of a non linear functional equation.* Nota IV (*) di GIOVANNI PROUSE, presentata dal Corrisp. L. AMERIO.

SUNTO. — Si danno le dimostrazioni dei teoremi 2, 3, 4 e 5 enunciati nel § 1 e si studia una particolare equazione a cui sono applicabili i risultati ottenuti.

4. Let us now give the proofs of Theorems 2, 3, 4 and 5.

Proof of Theorem 2: Consider the transformation S of the space V_2 in itself, defined by

$$Su(0) = u(T) \tag{4.1}$$

$u(\eta)$ being the solution in $[0, T]$ corresponding to the initial value $u(0)$.

By lemma 4, S is weakly continuous in V_2; moreover, by lemma 5, it maps each sphere of radius $R \geq \sqrt{K}$ in itself.

The transformation has therefore, by a theorem of Tychonoff (see for instance Dunford and Schwarz [1]) a fixed point, i.e. there exists a solution $\tilde{u}(\eta)$ such that

$$\tilde{u}(0) = \tilde{u}(T). \tag{4.2}$$

This solution is obviously periodic with period T.

(*) Pervenuta all'Accademia il 23 settembre 1967.

Proof of Theorem 3: Let $u_n(\eta)$ be the solution, defined for $\eta \geq -n$, satisfying the initial condition $u_n(-n) = 0$. Setting $u_n(\eta) = 0$ for $\eta < -n$ and

$$f_n(\eta) = \begin{cases} f(\eta) & \text{for } \eta \geq -n \\ 0 & \text{for } \eta < -n \end{cases} \tag{4.3}$$

the function $u_n(\eta)$ is obviously a solution in J of the equation

$$u_n'(\eta) + (A_1 + A_3)\, u_n(\eta) + BA_2\, u_n(\eta) = f_n(\eta)\,. \tag{4.4}$$

By lemma 6, it results

$$\begin{cases} \sup\limits_{\eta \in J} \| u_n(\eta) \|_{V_2} = M_1' \quad , \quad \sup\limits_{t \in J} \| u_n(t) \|_{L^2(0,1;W)} = M_2' \,, \\ \sup\limits_{t \in J} \| A_2 u_n(t) \|_{L^p(0,1;Y)} = M_3' \quad , \quad \sup\limits_{t \in J} \| A_2 u_n(t) \|_{H^\varepsilon(0,1;D(A_1^\varepsilon))} = M_4' \,. \end{cases} \tag{4.5}$$

Repeating, without any change, the procedure followed in Theorem 1 to prove the existence of a solution of the Cauchy problem, from relations (4.5) follows that it is possible to extract from $\{u_n(\eta)\}$ a subsequence (again denoted by $\{u_n(\eta)\}$) which converges, in the topologies introduced in (2.33), to a solution in J of (1.24), satisfying relations (1.27). The existence of a bounded solution in J is therefore proved.

Let us now show that, if also hypothesis XI) holds, this solution is unique.

Assume this is not so and let $v(t)$ be another solution $L^2(0\,,1\,;V_2)$-bounded in J. Setting $w(\eta) = u(\eta) - v(\eta)$, $w(\eta)$ is obviously a solution of the equation

$$w'(\eta) + (A_1 + A_3)\, w(\eta) + BA_2\, u(\eta) - BA_2\, v(\eta) = 0 \tag{4.6}$$

and satisfies the relation, analogous to (2.38),

$$\frac{1}{2} \langle w(\eta_2)\,, A_2 w(\eta_2) \rangle - \frac{1}{2} \langle w(\eta_1)\,, A_2 w(\eta_1) \rangle + \tag{4.7}$$

$$+ \int_{\eta_1}^{\eta_2} \langle (A_1 + A_3)\, w(\eta)\,, A_2 w(\eta) \rangle \, d\eta + \int_{\eta_1}^{\eta_2} \langle BA_2\, u(\eta) - BA_2\, v(\eta)\,, A_2 w(\eta) \rangle \, d\eta = 0\,.$$

By hypotheses III), VIII), XI), it results therefore, $\forall \eta \in J$ and $\forall \delta > 0$,

$$\| w(\eta - \delta) \|_{V_2}^2 \geq \| w(\eta) \|_{V_2}^2 + 2\alpha \int_{\eta-\delta}^{\eta} \| w(t) \|_W^2 \, dt - 2 c_5 \int_{\eta-\delta}^{\eta} \| w(t) \|_{D(A_2)}^2 \, dt \geq \tag{4.8}$$

$$\geq \| w(\eta) \|_{V_2}^2 + 2(\alpha - c_5 \gamma^2) \int_{\eta-\delta}^{\eta} \| w(t) \|_W^2 \, dt = \| w(\eta) \|_{V_2}^2 + \sigma_1 \int_{\eta-\delta}^{\eta} \| w(t) \|_W^2 \, dt,$$

being $\sigma_1 = 2(\alpha - c_5 \gamma^2) > 0$.

As the embedding of W in V_2 is continuous, we obtain, from (4.8),

$$\|w(\eta-\delta)\|_{V_2}^2 \geq \|w(\eta)\|_{V_2}^2 + \sigma_2 \int_{\eta-\delta}^{\eta} \|w(t)\|_{V_2}^2 \, dt . \tag{4.9}$$

Consequently, the function $\|w(\eta)\|_{V_2}$ is decreasing and from (4.9) it follows that

$$\|w(\eta-\delta)\|_{V_2}^2 \geq (1+\sigma_2\delta)\, \|w(\eta)\|_{V_2}^2 . \tag{4.10}$$

Hence

$$\lim_{t\to-\infty} \|w(t)\|_{L^2(0,1;V_2)} = +\infty , \tag{4.11}$$

which, for the hypotheses made, is absurd. $u(t)$ is then the only $L^2(0, 1; V_2)$–bounded solution in J.

By exactly the same procedure it can be proved that (1.28) holds. The theorem is then completely proved.

Proof of Theorem 4: In the proof of this and of the next theorem, we shall follow a procedure given by Amerio [2], [3] for linear functional equations.

In order that the theorem be proved, it will be sufficient, by Bochner's criterion, to show that it is possible to extract from any real sequence $\{l_n\}$ a subsequence $\{l'_n\}$ such that

$$\lim_{n\to\infty} \tilde{u}(t+l'_n) \underset{L^2(0,1;V_2)}{=} z(t) \tag{4.12}$$

uniformly in J.

In view of the hypotheses we have made, we can obviously assume that, uniformly in J

$$\lim_{n\to\infty}{}^{*} f(t+l'_n) \underset{L^{p'}(0,1;Y')}{=} g(t) . \tag{4.13}$$

Repeating, without any change, the procedure followed in Theorem 1, we find that it is possible to extract from $\{l_n\}$ a subsequence $\{l'_n\}$ such that, $\forall t \in J$,

$$\left\{\begin{aligned} &\lim_{n\to\infty}{}^{**} \tilde{u}(t+l'_n) \underset{L^\infty(0,1;V_2)}{=} z(t) , \\ &\lim_{n\to\infty}{}^{*} \tilde{u}(t+l'_n) \underset{L^2(0,1;W)}{=} z(t) , \\ &\lim_{n\to\infty}{}^{*} A_2 \tilde{u}(t+l'_n) \underset{L^p(0,1;Y)}{=} A_2 z(t) , \\ &\lim_{n\to\infty} A_2 \tilde{u}(t+l'_n) \underset{L^2(0,1;H)}{=} A_2 z(t) . \end{aligned}\right. \tag{4.14}$$

The function $z(\eta)$ is, moreover, a solution in J of the equation

$$z'(\eta) + (A_1 + A_3)\, z(\eta) + BA_2\, z(\eta) = g(\eta) . \tag{4.15}$$

Assume that (4.12) does not hold uniformly in J. There exist then a number $\sigma > 0$ and three sequences $\{t_n\}$, $\{\alpha'_n\} \subseteq \{l'_n\}$, $\{\alpha''_n\} \subseteq \{l'_n\}$ such that

$$\| \tilde{u}(t_n + \alpha'_n) - \tilde{u}(t_n + \alpha''_n) \|_{L^2(0,1;V_2)} \geq \sigma . \tag{4.16}$$

It is, on the other hand, possible to extract from $\{t_n + \alpha'_n\}$ and $\{t_n + \alpha''_n\}$ two subsequences (again denoted by $\{t_n + \alpha'_n\}$ and $\{t_n + \alpha''_n\}$) for which it results, in the various topologies introduced in (4.14),

$$\begin{cases} \lim\limits_{n \to \infty} \tilde{u}(t + t_n + \alpha'_n) = z_1(t) , \\ \lim\limits_{n \to \infty} \tilde{u}(t + t_n + \alpha''_n) = z_2(t) . \end{cases} \tag{4.17}$$

By the hypotheses made, it is however

$$\lim_{n \to \infty}{}^{*} f(t + t_n + \alpha'_n) = \lim_{n \to \infty}{}^{*} f(t + t_n + \alpha''_n) \underset{L^{p'}(0,1;Y')}{=} g(t) , \tag{4.18}$$

uniformly in J.

The functions $z_1(\eta)$ and $z_2(\eta)$ are therefore V_2–bounded solutions in J of equation (4.15). By theorem 3 it must then be $z_1(\eta) = z_2(\eta)$, which is in contrast with (4.16). Consequently, (4.12) holds uniformly in J and the theorem is proved.

Proof of Theorem 5: Let us prove at first that $\tilde{u}(\eta)$ is V_2–a.p.

Let $\{l_n\}$ be any real sequence; we shall show that it is possible to extract a subsequence $\{l'_n\} \subseteq \{l_n\}$ such that

$$\lim_{n \to \infty} \tilde{u}(\eta + l'_n) \underset{V_2}{=} z(\eta) \tag{4.19}$$

uniformly in J.

We may obviously assume that

$$\lim_{n \to \infty} f(t + l'_n) \underset{L^{p'}(0,1;Y')}{=} g(t) \tag{4.20}$$

and also, by Theorem 4,

$$\lim_{n \to \infty} \tilde{u}(t + l'_n) \underset{L^2(0,1;V_2)}{=} z(t) \tag{4.21}$$

uniformly in J.

Suppose that (4.19) does not hold uniformly in J; there exist then a number $\sigma > 0$ and three sequences $\{\eta_n\}$, $\{\alpha'_n\} \subseteq \{l'_n\}$, $\{\alpha''_n\} \subseteq \{l'_n\}$ such that

$$\| \tilde{u}(\eta_n + \alpha'_n) - \tilde{u}(\eta_n + \alpha''_n) \|_{V_2} \geq \sigma . \tag{4.22}$$

Setting $w_n(\eta) = \tilde{u}(\eta + \eta_n + \alpha'_n) - \tilde{u}(\eta + \eta_n + \alpha''_n)$, the function $w_n(\eta)$ satisfies the equation

$$\begin{aligned} w'_n(\eta) + (A_1 + A_3)\, w_n(\eta) + BA_2\, \tilde{u}(\eta + \eta_n + \alpha'_n) - \\ - BA_2\, \tilde{u}(\eta + \eta_n + \alpha''_n) = f(\eta + \eta_n + \alpha'_n) - f(\eta + \eta_n + \alpha''_n) \end{aligned} \tag{4.23}$$

and, $\forall \eta < 0$, the relation, analogous to (4.7),

$$\|w_n(\eta)\|^2_{V_2} \geq \|w_n(0)\|^2_{V_2} + \tag{4.24}$$

$$+ 2\int_\eta^0 \langle BA_2\,\tilde u(t+\eta_n+\alpha'_n) - BA_2\,\tilde u(t+\eta_n+\alpha''_n)\,,\,A_2\,w_n(t)\rangle\,dt +$$

$$+ 2\alpha\int_\eta^0 \|w_n(t)\|^2_W\,dt - 2\int_\eta^0 \langle f(t+\eta_n+\alpha'_n) - f(t+\eta_n+\alpha''_n)\,,\,A_2\,w_n(t)\rangle\,dt\,.$$

As $f(t)$ is $L^{p'}(0\,,1\,;Y')$–a.p. and $A_2 w_n(t)$ is $L^p(0\,,1\,;Y)$–bounded, it is possible to choose $n \geq n_\sigma$ so large that, when $-1 \leq \eta < 0$,

$$\left|\int_\eta^0 \langle f(t+\eta_n+\alpha'_n) - f(t+\eta_n+\alpha''_n)\,,\,A_2\,w_n(t)\rangle\,dt\right| \leq \frac{\sigma^2}{4}\,. \tag{4.25}$$

On the other hand, by (1.23) and condition XI),

$$\int_\eta^0 \langle BA_2\,\tilde u(t+\eta_n+\alpha'_n) - BA_2\,\tilde u(t+\eta_n+\alpha''_n)\,,\,A_2\,w_n(t)\rangle\,dt \geq \tag{4.26}$$

$$\geq -c_5\int_\eta^0 \|w_n(t)\|^2_{D(A_2)}\,dt \geq -c_5\gamma^2\int_\eta^0 \|w_n(t)\|^2_W\,dt \geq -\alpha\int_\eta^0 \|w_n(t)\|^2_W\,dt\,.$$

Hence, introducing (4.22), (4.25), (4.26) into (4.24),

$$\|w_n(\eta)\|^2_{V_2} \geq \frac{\sigma^2}{2} \qquad \forall \eta \in [-1\,,0]\,, \tag{4.27}$$

and, consequently,

$$\|w_n(-1)\|^2_{L^2(0,1\,;V_2)} = \int_{-1}^0 \|w_n(\eta)\|^2_{V_2}\,d\eta \geq \frac{\sigma^2}{2} \tag{4.28}$$

Relation (4.28) contradicts (4.21) and (4.19) holds therefore uniformly in J.

As $\tilde u(\eta)$ is V_2–continuous, it follows, by Bochner's criterion, that $\tilde u(\eta)$ is V_2–a.p.

Finally, we prove that $\tilde u(t)$ is $L^2(0\,,1\,;W)$–a.p.

Setting, in fact, $w_{jk}(\eta) = \tilde u(\eta+l'_j) - \tilde u(\eta+l'_k)$, it results (see (4.24))

$$\|w_{jk}(\eta)\|^2_{V_2} \geq \|w_{jk}(\eta+1)\|^2_{V_2} + \tag{4.29}$$

$$+ 2\int_\eta^{\eta+1} \langle BA_2\,\tilde u(t+l'_j) - BA_2\,\tilde u(t+l'_k)\,,\,A_2\,w_{jk}(t)\rangle\,dt +$$

$$+ 2\alpha\int_\eta^{\eta+1} \|w_{jk}(t)\|^2_W\,dt - 2\int_\eta^{\eta+1} \langle f(t+l'_j) - f(t+l'_k)\,,\,A_2\,w_{jk}(t)\rangle\,dt\,.$$

By (1.23) and hypothesis XI), it follows from (4.29), setting $\sigma_1 = 2(\alpha - c_5\gamma^2) > 0$,

$$(4.30) \qquad \sigma_1 \int_\eta^{\eta+1} \|w_{jk}(t)\|_W^2 \, dt \leq \|w_{jk}(\eta)\|_{V_2}^2 + 2 \int_\eta^{\eta+1} \langle f(t+l'_j) - f(t+l'_k), A_2 w_{jk}(t)\rangle \, dt .$$

By what has been proved above, the right hand side of (4.30) converges to zero, when $j, k \to \infty$, uniformly in J; $\tilde{u}(t)$ is therefore $L^2(0, 1; W)$–a.p. and the theorem is completely proved.

5. We now want to show that, under the assumptions made at the end of § 1, equation (1.33) satisfies hypotheses I)...X); the theorems and lemmas given in the preceding §§ will therefore hold for the solutions of this equation.

We observe, first of all, that the operators A_1 and A_2 defined by (1.29), (1.30) are obviously linear, positive, self adjoint and permutable (having constant coefficients) and therefore satisfy condition II). Moreover, if Γ is of class C^2, it results, by a theorem of Nirenberg [4],

$$(5.1) \qquad D(A_1) = D(E) = H^2(\Omega) \cap H_0^1(\Omega) .$$

Hence, $\forall \alpha \in [0, 1]$,

$$(5.2) \qquad D(A_1^\alpha) = D(E^\alpha) .$$

As $A_2 = E^\sigma$, it is then

$$(5.3) \qquad D(A_1^\sigma) = D(A_2)$$

so that hypothesis I) is verified.

Being, in the case we are now considering, $V_1 = D(A_1^{1/2}) = H_0^1(\Omega)$, $V_2 = D(A_2^{1/2}) = D(A_1^{\sigma/2})$, $W = D(A_1^{(1+\sigma)/2})$, $A_1 W = D(A_1^{(\sigma-1)/2})$, $A_2 W = D(A_1^{(1-\sigma)/2})$, it is obviously, by the second of (1.29),

$$(5.4) \qquad A_3 W \subset A_3 H_0^1(\Omega) \subset L^2(\Omega) \subset A_1 W ,$$

that is hypothesis IV).

Setting $\nu = \sup_{\substack{x \in \Omega \\ j=0,1,\cdots,m}} |a_j(x)|$, it results, by (1.29), (1.30),

$$(5.5) \qquad (A_1 v, A_2 v)_{L^2(\Omega)} + (A_3 v, A_2 v)_{L^2(\Omega)} \geq \|v\|^2_{D(A_1^{(1+\sigma)/2})} - \|A_3 v\|_{L^2(\Omega)} \|v\|_{D(A_1^\sigma)} .$$

Hence, if c is the embedding constant of $D(A_1^{(1+\sigma)/2})$ in $D(A_1^\sigma)$,

$$(5.6) \qquad ((A_1 + A_3) v, A_2 v)_{L^2(\Omega)} \geq \|v\|^2_{D(A_1^{(1+\sigma)/2})} - c \|A_3 v\|_{L^2(\Omega)} \|v\|_{D(A_1^{(1+\sigma)/2})} .$$

For hypothesis III) to be verified, it is therefore sufficient that

$$(5.7) \qquad \|A_3 v\|_{L^2(\Omega)} \leq \frac{\delta}{c} \|v\|_{D(A_1^{(1+\sigma)/2})} \qquad \text{with } \delta < 1,$$

which is obviously true then ν is small enough.

Setting $Y = L^p(\Omega)$, from the hypotheses made on the function $\beta(\xi)$ it follows that conditions V), VI), VIII) are satisfied.

Let us prove that also condition VII) holds. Let $\{v_n(x,\eta)\}$ be a sequence such that, setting $Q = \Omega \times [0, T]$,

$$\lim_{n\to\infty}{}^*_{L^p(Q)} v_n = v \quad , \quad \lim_{n\to\infty}{}_{L^2(Q)} v_n = v . \tag{5.8}$$

From (1.31) and the first of (5.8) it follows that it is possible to extract from $\{v_n\}$ a subsequence (again denoted by $\{v_n\}$) such that

$$\lim_{n\to\infty}{}^*_{L^{p'}(Q)} \beta(v_n) = \psi \qquad \left(\frac{1}{p} + \frac{1}{p'} = 1\right). \tag{5.9}$$

We shall show that

$$\psi = \beta(v). \tag{5.10}$$

Let us observe that, by the second of (5.8), it is possible to extract from $\{v_n\}$ a subsequence (again denoted by $\{v_n\}$) such that

$$\lim_{n\to\infty} v_n(x,\eta) = v(x,\eta) \qquad \text{almost everywhere in } Q. \tag{5.11}$$

Hence

$$\lim_{n\to\infty} \beta(v_n(x,\eta)) = \beta(v(x,\eta)) \qquad \text{almost everywhere in } Q. \tag{5.12}$$

If ε is an arbitrary positive number, there exists a closed set Q_ε, with $m(Q - Q_\varepsilon) < \varepsilon$, such that in Q_ε the convergence is uniform and all the functions v_n are continuous. It is then, $\forall h \in L^p(Q)$, with $h = 0$ in $Q - Q_\varepsilon$,

$$\lim_{n\to\infty} \int_Q \beta(v_n(x,\eta))\, h(x,\eta)\, dQ = \int_{Q_\varepsilon} \psi(x,\eta)\, h(x,\eta)\, dQ = \tag{5.13}$$
$$= \int_{Q_\varepsilon} \beta(v(x,\eta))\, h(x,\eta)\, dQ .$$

Consequently, being h arbitrary, $\psi(x,\eta) = \beta(v(x,\eta))$ almost everywhere in Q_ε and hence also in Q. Relation (5.10) is therefore proved and hypothesis VII) verified.

Let us finally show that also conditions IX) and X) hold.

If Γ is of class C^{2s}, with $s = \left[\frac{m(p-2)}{4p}\right] + 1$, it follows from a theorem of Nirenberg [4] and an embedding theorem of Sobolev (see Gagliardo [5]) that

$$D(A_1^s) \subset H^{2s}(\Omega) \cap H_0^1(\Omega) \subset L^p(\Omega). \tag{5.14}$$

Being $D(A_1^s)$ dense in $L^p(\Omega)$, we obtain, from (5.14),

$$L^{p'}(\Omega) \subset D(A_1^s)', \tag{5.15}$$

i.e. hypothesis X). On the other hand, $D(A_1^{(1-\sigma)/2}) \cap L^p(\Omega)$ is separable, as it is contained in $H^{1-\sigma}(\Omega) \cap L^p(\Omega)$, which is separable, being isometric to a direct sum of $L^q(\Omega)$ spaces, with various separable measures. Moreover, being $D(A_1^{(1-\sigma)/2}) \cap L^p(\Omega) \supset H_0^1(\Omega) \cap L^p(\Omega) \supset \mathfrak{D}(\Omega)$ and $\mathfrak{D}(\Omega)$ dense in $D(A_1^{(1-\sigma)/2})$ and in $L^p(\Omega)$, the space $D(A_1^{(1-\sigma)/2}) \cap L^p(\Omega)$ is dense in $D(A_1^{(1-\sigma)/2})$ and in $L^p(\Omega)$. Also hypothesis IX) is therefore verified.

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