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Biorthonormal systems in vector spaces

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Analisi matematica. — *Biorthonormal systems in vector spaces* (*). Nota di MENDEL DAVID e JACOB STEINBERG, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota viene definita la nozione di una serie di Appell in uno spazio vettoriale, la quale costituisce una generalizzazione delle serie di polinomi di Appell. Una serie di Appell vettoriale viene studiata in connessione con una sequenza di funzionali lineari, che la completano ad un sistema biortonormale.

1. INTRODUCTION.

Let L be a (complex) vector space and let A be a linear operator mapping L into itself. Let us suppose that there exists in L a sequence $x_0 , x_1 , x_2 \cdots$ such that $x_0 \neq 0$, $Ax_i = x_{i-1}$ for $i = 1 , 2 , 3 , \cdots$ and $Ax_0 = 0$.

By means of a sequence of complex numbers $a_0 , a_1 , a_2 , \cdots$ with $a_0 = 1$ one may construct the set:

$$y_i = \sum_{j=0}^{i} a_j x_{i-j} \qquad i = 0 , 1 , 2 \cdots$$

This will be called in the sequel an Appell set of vectors relative to the sequence $\{x_n\}$ and the operator A. When a_i are the coefficients of the power series representation of a complex variable function $F(u)$ analytic about the origin we say that the Appell set y_i is generated by $F(u)$.

In the case in which L is a space of infinite differentiable functions, A is the differentiation operator and $x_i = \frac{t^i}{i!}$ the notion of Appell set of vectors coincides with the well known notion of Appell set of polynomials.

A more general example is that of the polynomials of type k which is obtained when A is an operator of the form

$$A = \sum_{n=1}^{\infty} P_n(t) \frac{d^n}{dt^n}$$

where $P_n(t)$ is a polynomial of the form $l_{n0} + l_{n1} t + \cdots + l_{nn-1} t^{n-1}$ for $n = 1 , 2 , \cdots$ and the following two conditions are fulfilled:

(a) $nl_{10} + n(n-1) l_{21} + \cdots + n! \, l_{n,n-1} \neq 0$

(b) The degrees k_n of P_n satisfy $\max_n k_n = k < \infty$.

Moreover, it is supposed that L is a space of infinite differentiable functions in which A is defined and x_i is the (unique) basic set of type k (see [1] p. 592).

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Some properties of Appell sets and sets of type zero of polynomials were discovered in [2] and [3]. We shall deal here with the general case of the Appell set of vectors and shall prove two theorems.

Theorem 1 is analogous to theorem 3 in [2] and theorem 6 in [3]. Theorem 2 is a generalization of theorem 6 in [2].

2. Before we state the first theorem we recall the definition of a biorthonormal system.

A sequence z_n, $n = 0, 1, 2, \cdots$ in L and a sequence of linear functionals $\{g_n\}$ $n = 0, 1, 2, \cdots$ in the dual space of L, L^* form a biorthonormal system if

$$g_n(z_m) = \delta_{nm} \qquad n, m = 0, 1, 2 \cdots$$

where

$$\delta_{nm} = \begin{cases} 0 & \text{for } m \neq n \\ 1 & \text{for } m = n \end{cases}$$

THEOREM 1. Let $\{y_i\}$ be an Appell set relative to x_i and the operator A. Also let L_1 be the vector subspace of L formed by the finite linear combinations of x_i. Then there exists at least one functional f in L^* such that the sequence $\{(A^*)^n f\}$ completes the sequence $\{y_n\}$ to a biorthonormal system. The functionals f with this property have a unique restriction f_0 on L_1. If $\{y_n\}$ is generated by a function $F(u)$ analytic about the origin with $F(0) = 1$ then $f_0(x_i)$, $i = 0, 1, 2, \cdots$ are the coefficients of the power series representation of $F^{-1}(u)$.

Proof: Let f be a linear functional in L^* with $f(x_0) = 1$. Since $Ax_n = x_{n-1}$ for $n = 1, 2, \cdots$ and $Ax_0 = 0$ one obtains easily $Ay_n = y_{n-1}$ for $n = 1, 2, \cdots$ and $Ay_0 = 0$. Therefore, for $n > m$, $A^n y_m = 0$. Hence

$$(A^*)^n f(y_m) = f(A^n y_m) = 0$$

in this case. For $n = m$

$$f(A^n y_m) = f(y_0) = f(x_0) = 1$$

If $n < m$

$$f(A^n y_m) = f(y_{m-n}) = \sum_{j=0}^{m-n} a_j f(x_{m-n-j}) .$$

But the system

$$\sum_{j=0}^{k} a_j t_{k-j} = 0 \tag{1}$$

where $t_0 = 1$, has a unique solution $t_1, t_2, t_3 \cdots$

Therefore, in order to complete the proof of the first two parts of the theorem we must show that there exist linear functionals f in L^* satisfying $f(x_i) = t_i$ for $i = 0, 1, 2, \cdots$ and that the restrictions of all functionals f on L_1 coincide. To prove this it is enough to show that the sequence $\{x_i\}$ is a part of a Hamel basis of L. But, by Zorn's lemma, this will follow if we show that every finite set of vectors x_i is linearly independent. Since $x_0 \neq 0$ and $A^n x_n = x_0$ for $n = 0, 1, 2, \cdots$ every set containing one vector is linearly independent. Next, we will suppose that every set of $n-1$ vectors is linearly independent and we will show that every set of n vectors $x_{i_1}, x_{i_2}, \cdots, x_{i_n}$ where $i_1 < i_2 < \cdots < i_n$ has the same property. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be n numbers such that

$$\alpha_1 x_{i_1} + \alpha_2 x_{i_2} + \cdots + \alpha_n x_{i_n} = 0.$$

An application of A^{i_n} yields $\alpha_n = 0$. This fact together with the assumption that $x_{i_1}, x_{i_2} \cdots x_{i_{n-1}}$ are linearly independent imply the linear independence of $x_{i_1}, x_{i_2}, \cdots, x_{i_n}$. The first two parts of the theorem are proved.

If a_j are the coefficients of the power series representation of an analytic function $F(u)$ about the origin then the Cauchy multiplication formula and (1) show that $t_i = f_0(x_i)$, $i = 0, 1, 2, \cdots$ must be the coefficients of the power series representation of $F^{-1}(u)$ (which is analytic about the origin since $F(u)$ is analytic and $F(0) = 1$). Theorem 1 is proved.

Example 1. Let L be the vector space of infinitely differentiable functions in $(-\infty, +\infty)$; $A = \frac{d}{dt} - t$ and $x_k = \frac{1}{k!} t^k e^{\frac{t^2}{2}}$.

Suppose that y_k is an Appell set relative to this sequence, generated by a function $F(u)$. Suppose also that $F^{-1}(iy)$ as a real variable function and its derivatives are defined, for $-\infty < y < \infty$, and vanish more rapidly than any negative power of y at infinity. In the present case, as in the case treated in [2], section 3, the functional f_0 has an integral form

$$f_0(x) = \int_{-\infty}^{+\infty} p(t)\, x(t)\, dt$$

where

$$p(t) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{-t\left(u + \frac{t}{2}\right)} F^{-1}(u)\, du = \frac{1}{2\pi} \int_{-\infty}^{-\infty} e^{-t\left(iy + \frac{t}{2}\right)} F^{-1}(iy)\, dy .$$

In fact, by Fourier's integral theorem we have

$$F^{-1}(iy) = \int_{-\infty}^{+\infty} e^{t\left(iy + \frac{t}{2}\right)} p(x)\, dt$$

and therefore

$$\frac{1}{k!} \frac{d^k}{du^k} F^{-1}(u) \Big|_{u=0} = f_0(x_k), \qquad \text{for } k = 0, 1, 2, \cdots.$$

We remark also that the sequence of linear functionals g_n is here

$$g_n(x) = (A^*)^n f_0(x) = \int_{-\infty}^{+\infty} \left[\left(-\frac{d}{dt} - t\right)^n p(t)\right] x(t)\, dt .$$

Example 2. Let $x_n = O_n(t)$ where t is complex $t \neq 0$, and

$$O_n(t) = \frac{1}{2} \int_0^\infty e^{-vt} [(v + \sqrt{v^2+1})^n + (v - \sqrt{v^2+1})^n] \, dv ,$$

which is a polynomial of degree $n+1$ in t (the Laplace integral being convergent for $\mathrm{R}e\, t > 0$).

It is well known that, if $J_m(t)$ is Bessel's function of order m,

$$\int J_m(t) \, O_n(t) \, dt = 0 ; \qquad m \neq n ,$$

on a circle around the origin (see [4], pp. 262–264).

Let

$$K(s, t) = i\pi \sum_{n=0}^{\infty} J_{n+1}(s) \, O_n(t) ,$$

then the series converges absolutely and uniformly for $|t| \geq |s| + \varepsilon$ $(\varepsilon > 0)$, the proof being the same as for the series with J_n replacing J_{n+1} (see [4], pp. 263–264). Thus

$$\int_{|s|=|t|-\varepsilon} K(s, t) \, O_{n+1}(s) \, ds = O_n(t) ; \qquad (n = 0, 1, \cdots)$$

and

$$\int_{|s|=|t|-\varepsilon} K(s, t) \, O_0(s) \, ds = \int_{|s|=|t|-\varepsilon} \frac{K(s, t)}{s} \, ds = K(0, t) = 0 .$$

Thus A is an integral operator with kernel $K(s, t)$, and the adjoint A^* defined by

$$A^* x(t) = \int_{|t|=|s|+\varepsilon} K(s, t) \, x(t) \, dt .$$

We have indeed $A^* J_n = J_{n+1}$, $(n = 0, 1, \cdots)$.

In the sequel we will need the following definition:

Let $\{x_i\}$ be a sequence in L with $x_0 \neq 0$ and A a linear operator such that $Ax_0 = 0$ and $Ax_i = x_{i-1}$ for $i = 1, 2, 3, \cdots$. A sequence of vectors of the form

$$(2) \qquad y_i = \sum_{j=0}^{i} b_{ij} \, x_j \qquad i = 0, 1, 2, \cdots$$

is called a standard set relative to the set $\{x_i\}$ and the operator A.

An Appell set is obviously a special case of a standard set. Moreover, we remark that this notion of standard set generalizes the notion of standard set of polynomials introduced in [2] p. 197.

Now we will prove the following

THEOREM 2. Let $\{x_i\}$ and A be as above and let f be a linear functional in L^* satisfying $f(x_0) = 1$. Then there exists a unique standard set relative to x_i and A, namely an Appell set y_i which completes the sequence $(A^*)^n f$ to a biorthonormal system. Moreover, if $f(x_i)$ are the coefficients of the power series representing an analytic function $F(u)$ about the origin then the Appell set is generated by $F^{-1}(u)$.

Proof: First we will show that there exists a unique Appell set $\{y_i\}$ such that

$$(A^*)^n f(y_m) = \delta_{nm} \qquad n, m = 0, 1, 2, \cdots \tag{3}$$

We see from the proof of theorem 1 that (1) holds if and only if the sequence $\{a_i\}$ corresponding to $\{y_i\}$ satisfies the system

$$\sum_{j=0}^{k} a_j f(x_{k-j}) = 0 .$$

Since this system has a unique solution $a_0, a_1, a_2, \cdots$ with $a_0 = 1$, the existence and the uniqueness of the Appell set $\{y_i\}$ is proved.

The proof of the last part of this theorem is analogous to that of the last part of theorem 1. It remains therefore to show that every standard set $\{y_i\}$ satisfying (3) is an Appell set. This means that we have to prove that the coefficients b_{ij} from (2) satisfy the conditions

$$b_{ii} = 1 \qquad \text{for} \quad i = 0, 1, 2, \cdots \tag{4}$$

$$b_{i,i-k} = b_{i+1,i+1-k} \qquad \text{for} \quad i = 1, 2, \cdots; k = 1, 2, \cdots \tag{5}$$

(4) follows substituting in (3) $m = n = i$. In fact, this substitution gives

$$b_{ii} = b_{ii} f(x_0) = f(b_{ii} x_0) = f(A^i y_i) = (A^*)^i f(y_i) = 1 .$$

Let us prove now (5) for $k = 1$. Subtracting the equality which is obtained by substituting $n = i - 1$ and $m = i$ in (3) from that obtained for $n = i$ and $m = i + 1$ we get

$$f(A^i y_{i+1} - A^{i-1} y_i) = 0 .$$

Hence

$$f[(b_{i+1,i+1} x_1 + b_{i+1,i} x_0) - (b_{ii} x_1 + b_{i,i-1} x_0)] = 0.$$

Since $b_{i+1,i+1} = b_{ii} = 1$ and $f(x_0) = 1$ this equality implies $b_{i+1,i} = b_{i,i-1}$.

Let us now suppose that (5) holds for $k = 2, \cdots, p; p < i$. We prove that this implies (5) with $k = p + 1$. Subtracting the equality obtained by the substitution $n = i - p - 1, m = i$ from that obtained for $n = i - p$, $m = i + 1$ we get

$$f(A^{i-p} y_{i+1} - A^{i-p-1} y_i) = 0$$

This implies

$$f((b_{i+1,i+1}x_{p+1}+b_{i+1,i}x_p+\cdots+b_{i+1,i-p+1}x_1+b_{i+1,i-p}x_0)-$$
$$-(b_{ii}x_{p+1}+b_{i,i-1}x_p+\cdots+b_{i,i-p}x_1+b_{i,i-1-p}x_0))=0.$$

Since (4) and (5) with $k=1,2,\cdots,p-1$ hold, this equality may be written in the form

$$f(b_{i+1,i-p}x_0-b_{i,i-1-p}x_0)=0$$

Hence $b_{i+1,i-p}=b_{i,i-1-p}$. Theorem 2 is proved.

Remark. We see that the existence of the sequence x_i implies the existence of an infinite number of Appell sets relative to the operator A. Furthermore, one can show that there exists a unique standard set z_i relative to x_i and A, such that

$$b_{00}=1 \quad , \quad b_{ii}\neq 0 \quad \text{and} \quad b_{i0}=0 \quad \text{for} \quad i=1,2,\cdots$$

and $Bz_i=z_{i-1}$, $i=1,2,\cdots$, $Bz_0=0$ where B is an operator of the form

$$B=\sum_{i=1}^{\infty}c_i A^i, \qquad c_1\neq 0.$$

Hence, the existence of the set $\{x_i\}$ implies also the existence of an infinite number of Appell sets relative to B and to $\{z_i\}$.

References.

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