Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Structure Theory in s-d—Rings. Nota III

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **43** (1967), n.6, p. 477–479.

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<http://www.bdim.eu/item?id=RLINA_1967_8_43_6_477_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1967.

Matematica. — *Structure Theory in s–d–Rings*. Nota III di ESAYAS GEORGE KUNDERT, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si completa l'indagine svolta in due Note precedenti [1, 2], effettuando uno studio della struttura moltiplicativa dell'insieme degli « inteals » di un *s–d*–anello.

We return in this note to investigate further the inteal structure on a *s–d*–ring [1]. We heavily exploit here the existence of the σ and τ homomorphisms in *s–d*–rings (see note II, [2]) to obtain:

(1) a factorization theorem for inteals (see theorem 1).

(2) a characterization of the inteals in the case where the ring of constants is a field (theorem 2).

(3) implications linking up the notions of prime and multiplicatively irreducible inteals in a *s–d*–ring with the same notions for ideals in the ring of constants (theorem 3).

Let $\mathfrak{A}$ be a *s–d*–ring [1], and let A be an ideal in $\mathfrak{A}$.

Definition: Let $\bar{\mathrm{A}} = \sigma(\mathrm{A}) \cdot \tau(\mathrm{A})$ be called the norm of A.

It is clear that:

(1) $\bar{\mathrm{A}}$ is an ideal in R,

(2) $\overline{\mathrm{AB}} = \bar{\mathrm{A}} \cdot \bar{\mathrm{B}}$,

(3) $\bar{\mathfrak{A}} = \mathrm{R}$.

Definition: An ideal N of $\mathfrak{A}$ is called a null-ideal if $\bar{\mathrm{N}} = (0)$.

If N is a null-ideal $\Rightarrow \mathrm{N} \cdot \mathrm{A}$ is also a null-ideal for any ideal A in $\mathfrak{A}$.

LEMMA 1. *If* A *is an inteal in* $\mathfrak{A} \Rightarrow (\mathrm{A} \cap \mathrm{R})^2 \subseteq \bar{\mathrm{A}} \subseteq \mathrm{A} \cap \mathrm{R}$.

Proof: (1) To prove $\bar{\mathrm{A}} \subseteq \mathrm{A} \cap \mathrm{R}$ we only have to show that from $a, b \in \mathrm{A} \Rightarrow \tau(a) \cdot \sigma(b) \in \mathrm{A}$. By $\equiv$ in this proof we always mean $\equiv$ mod A. Let

$$a = \sum_{j=0}^{n} \alpha_j x_j \equiv 0 \Rightarrow s^{(n-i)}(x_i\, a) \equiv 0, \qquad 0 \leq i \leq n$$

$$\Rightarrow \left\{ s^{(n-i)}(x_i\, a) = \left(\sum_j \alpha_j \alpha_{ij}^{i}\right) x_n + \left(\sum_j \alpha_j \alpha_{ij}^{i+1}\right) x_{n+1} + \cdots + \alpha_n \alpha_{in}^{i+n} x_{2n} \equiv 0 \right\}.$$

$$0 \leq i \leq n.$$

This is a system of $n+1$ homogeneous equations in $x_n, x_{n+1}, \cdots, x_{2n}$. [We used here the fact that $\alpha_{ij}^{s} = 0$ for $s < i$; in the case of a *s–d*–ring formula (3) in paragraph 3 [2] actually turns into $\alpha_{ij}^{s} = (-1)^{s-i+j} \binom{j}{s-i} \binom{s}{j}$].

By using formula (3) [2], we are able to eliminate successively $x_{2n}, x_{2n-1}, \cdots, x_{n-1}$ and we end up with: $x_n \cdot \tau(a) \equiv 0$ and therefore also $s^{(k)}(x_n \cdot \tau(a)) = x_{n+k} \cdot \tau(a) \equiv 0$ for $k \geq 0$.

(*) Nella seduta del 9 dicembre 1967.

Now let $b = \sum_{j=0}^{m} \beta_j x_j \equiv 0$. If $m = 0$ we have $b = \beta_0 = \sigma(b)$ and $\sigma(b) \cdot \tau(a) \equiv 0$. If $m \geq 1$ then $\tau(a) \cdot b = \tau(a) \cdot \sigma(b) + \sum_{j=1}^{m} \beta_j \tau(a) \cdot x_j \equiv 0$. Because $\tau(a) \cdot x_j \equiv 0$ for $j \geq n$ we may assume that $m = n - 1$ and to obtain $\tau(a) \cdot \sigma(b) \equiv 0$ it will suffice to prove that $\beta_j \cdot \tau(a) \cdot x_j \equiv 0$ for $1 \leq j \leq n - 1$. Now $\tau(a)\, s^{(n-1)}(b) = \beta_0 \tau(a) x_{n-1} + \beta_1 \tau(a) x_n + \cdots \equiv \beta_0 \tau(a) x_{n-1} \equiv 0$. Assume that we already proved $\beta_k \tau(a) x_{n-1} \equiv 0$ for $k < i \leq n - 1$. Take $\tau(a)\, s^{(n-i-1)}(x_i b) = \sum_{k=0}^{n-1} \tau(a) \beta_k \sum_s \alpha_{ki}^s x_{s+n-i-1} \equiv \tau(a) \beta_i x_{n-1} \equiv 0$. (We used here again that $\alpha_{ki}^s = 0$ for $s < i$ and also that $\alpha_{ii}^i = \pm 1$ for s–d–rings). Therefore by induction: $\tau(a) \beta_i x_{n-1} \equiv 0$ for all $i \leq n - 1$ and specifically $\tau(a) \beta_{n-1} x_{n-1} \equiv 0$. We can now assume that $m = n - 2$ and repeat the above argument to get $\tau(a) \beta_i x_{n-2} \equiv 0$ for all $i \leq n - 2$, and specially $\tau(a) \beta_{n-2} x_{n-2} \equiv 0$. Repeat for $m = n - 3, \cdots, 1$. We get $\tau(a) \beta_j x_j \equiv 0$ for all $j \geq 1$.

(2) To prove $(A \cap R)^2 \subseteq \bar{A}$ let $\alpha \in (A \cap R)^2 \Rightarrow \alpha = \Sigma \alpha_i \beta_i$; $\alpha_i, \beta_i \in A \cap R$. Since $\alpha_i, \beta_i \in A \Rightarrow \alpha_i \beta_i = \sigma(\alpha_i) \tau(\beta_i) \in \bar{A} \Rightarrow \alpha \in \bar{A}$.

LEMMA 2. *If* A *is an inteal in* $\mathfrak{A}$ *and* $\bar{A} = R \Rightarrow A = \mathfrak{A}$.

Proof: By Lemma 1, we have $R = \bar{A} \subset A \Rightarrow A = \mathfrak{A}$.

Definition: An inteal in $\mathfrak{A}$ is called proper if it is not a null-inteal and not $\mathfrak{A}$.

Definitions: (1) A factorization of a proper inteal $A = \prod_{i=1}^{n} A_i$ is called proper if all A_i are proper inteals in $\mathfrak{A}$.

(2) A proper refinement of a proper factorization is a proper factorization $A = \prod_{j=1}^{m} A'_j$ such that there exist indices $j_0 = 1 < j_1 < \cdots < j_n = m$ with $m > n$ such that $A_i = \prod_{j=j_i-1}^{j_i} A'_j$.

(3) An inteal is called multiplicatively irreducible if A has no proper refinement.

(4) A refinement chain of an inteal is a sequence of proper factorizations with each term being a proper refinement of the preceding one.

(5) A refinement chain is said to terminate if there is a last term which cannot be properly refined. Note: This last term must have multiplicatively irreducible factors only.

Definition: We say that $\mathfrak{A}$ has property I if every refinement chain of every proper inteal terminates.

We also define all the above notions for the ring of constants R by replacing " inteal " by " ideal ". It is well known that if R is a noetherian integral domain then R has property I. For an interesting theorem in connection with these definitions see [3].

THEOREM 1. *Let* $\mathfrak{A}$ *be a s–d–ring. If the ring of constants has property* I $\Rightarrow$ $\mathfrak{A}$ *has property* I $\Rightarrow$ *every proper inteal factors into a finite number of multiplicatively irreducible inteals.*

Proof: From Lemma 2 follows that if $A = \prod_{i=1}^{n} A_i$ is a proper factorization in $\mathfrak{A}$ then $\bar{A} = \prod_{i=1}^{n} \bar{A}_i$ is also a proper factorization in R. Taking the norm of a non terminating refinement chain of A, we would get a non terminating refinement chain of $\bar{A}$ in R which is a contradiction with property I in R. $\mathfrak{A}$ must therefore have property I.

THEOREM 2. *If the ring of constants* R *of a s–d–ring is a field* $\Rightarrow$ *the only inteals in* $\mathfrak{A}$ *are the null-inteals and* $\mathfrak{A}$ *itself*.

Proof: Let A be a proper inteal in $\mathfrak{A}$. By Lemma $2 \Rightarrow \bar{A}$ is a proper ideal in R. Contradiction!

Definition: R is said to have property II iff $\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{a}$ for any two proper ideals in R. If R has property I $\Rightarrow$ R has property II.

THEOREM 3. *Let* $\mathfrak{A}$ *be an s–d–ring*, A *an inteal in* $\mathfrak{A}$. *The following implications hold true*:

(1) $\bar{A}$ multiplicatively irreducible in $R \Rightarrow A$ multiplicatively irreducible in $\mathfrak{A}$.

(2) If R has property II and A is a proper prime inteal in $\mathfrak{A} \Rightarrow A$ is multiplicatively irreducible in $\mathfrak{A}$.

(3) If R is the ring of integers and A is a prime inteal in $\mathfrak{A} \Rightarrow \bar{A} = A \cap R$ and $\bar{A}$ is a prime ideal in R.

Proof: (1) Suppose $A = B \cdot C$, $B \neq \mathfrak{A}$, $C \neq \mathfrak{A} \Rightarrow \bar{A} = \bar{B} \cdot \bar{C}$ and by lemma $2 \Rightarrow \bar{B} \neq R$, $\bar{C} \neq R \Rightarrow \bar{A}$ multiplicatively reducible. Contradiction!

(2) Suppose $A = B \cdot C$, $B \neq \mathfrak{A}$, $C \neq \mathfrak{A} \Rightarrow B \supset A$ and $C \supset A$ because if say $B = A \Rightarrow \bar{A} = \bar{A} \cdot \bar{C}$ and by Lemma 2, $\bar{A}$, $\bar{C}$ are proper ideals. Contradiction with property II. Let $b \in B$, $b \notin A$ and $c \in C$, $c \notin A \Rightarrow b \cdot c \in A \Rightarrow A$ not prime!

(3) A prime $\Rightarrow A \cap R$ prime $= (p)$. By Lemma 1 we have $(p)^2 \subseteq \bar{A} \subseteq (p)$. So either $\bar{A} = (p)$ in which case we are finished, or $\bar{A} = (p)^2$. We show that if $\bar{A} = (p)^2 \Rightarrow A$ not prime. Since $p \in A \Rightarrow p \cdot x_p \in A$, but in a *s–d*–ring we have $x_1 \cdot x_{p-1} = p \cdot x_p - (p-1) \cdot x_{p-1}$ or $(x_1 + p - 1) \cdot x_{p-1} \in A$. Now if A were prime it would follow that $x_{p-1} \in A$ or $x_1 + p - 1 \in A$. In the first case, we would have: $\tau(x_{p-1}) \, \sigma(p) = 1 \cdot p \in \bar{A}$. Contradiction! In the second case we would have: $\sigma(x_1 + p - 1) \, \tau(p) = (p-1) \, p \in \bar{A}$, but $p^2 \in \bar{A} \Rightarrow p \in \bar{A}$. Contradiction!

Remark: The implications in Theorem 3 are not reversible. This can be easily shown by counter-examples.

LITERATURE.

[1] E. G. KUNDERT, *Structure Theory in s–d–Rings*. Note I, «Accademia Nazionale dei Lincei», Ser. VIII, vol. XLI, fasc. 5, November, 1966.

[2] E. G. KUNDERT, *Structure Theory in s–d–Rings*. Note II, «Accademia Nazionale dei Lincei», Ser. VIII, vol. XLIII, fasc. 5, Novembre, 1967.

[3] H. S. BUTTS, *Unique Factorization of Ideals into Nonfactorable Ideals*, «Proc. of the Amer. Math. Soc.», vol. 15, No. 1, February, 1964.