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Classe Scienze Fisiche Matematiche Naturali

# Rendiconti

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## Periodic or almost-periodic solutions of a non linear functional equation. Nota III

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# NOTE PRESENTATE DA SOCI

**Analisi matematica.** — *Periodic or almost-periodic solutions of a non linear functional equation.* Nota III (*) di Giovanni Prouse, presentata dal Corrisp. L. Amerio.

Riassunto. — Si dimostrano alcuni lemmi che verranno utilizzati nel § 4 per la dimostrazione dei teoremi di periodicità o quasi-periodicità.

3. In this § we shall prove some further lemmas, which will be utilized in the proofs of the theorems regarding periodic, bounded or a.p. solutions.

Lemma 4: *Assume that the hypotheses of theorem* 1 *hold. Then the solution of the Cauchy problem depends continuously,* $\forall \eta \in [0, T]$, *on the initial data, in the weak topology of* $V_2$.

Let $\{u_{0,n}\}$ be a sequence of elements $\in V_2$ and let $\{u_n(\eta)\}$ be the sequence of the solutions in $[0, T]$ satisfying the initial condition

$$u_n(0) = u_{0,n}. \tag{3.1}$$

Assume, moreover, that

$$\lim_{n \to \infty}{}^{*} u_{0,n} \underset{V_2}{=} u_0. \tag{3.2}$$

In order that the lemma be proved, it will be sufficient to show that it is possible to extract from $\{u_n(\eta)\}$ a subsequence (which we shall again call $\{u_n(\eta)\}$) such that, $\forall \eta \in [0, T]$, it is

$$\lim_{n \to \infty}{}^{*} u_n(\eta) \underset{V_2}{=} u(\eta), \tag{3.3}$$

$u(\eta)$ being the solution that satisfies the initial condition

$$u(0) = u_0. \tag{3.4}$$

Let us observe, first of all, that, by (3.2),

$$\|u_{0,n}\|_{V_2} \le K_0. \tag{3.5}$$

It results then, applying (2.5) to the interval $[0, \eta]$ $(0 < \eta \le T)$

$$\sup_{\eta \in [0,T]} \|u_n(\eta)\|_{V_2} \le K_7 \quad , \quad \int_0^T \|u_n(\eta)\|_W^2 \, d\eta \le K_8, \tag{3.6}$$

$$\int_0^T \|A_2 u_n(\eta)\|_Y^p \, d\eta \le K_9.$$

(*) Pervenuta all'Accademia il 23 settembre 1967.

Repeating, without any change, the procedure followed in theorem 1, we then see that it is possible to extract from $\{u_n(\eta)\}$ a subsequence (again called $\{u_n(\eta)\}$) which converges, in the topologies introduced in (2.33), to a solution in $[0, T]$ of (1.24); we shall denote this solution by $u(\eta)$.

We now prove that, $\forall \eta \in [0, T]$.

$$\lim_{n\to\infty}{}^{*} u_n(\eta) \underset{V_2}{=} u(\eta). \tag{3.7}$$

Let $h$ be any element of the basis $\{h_j\}$ introduced in theorem 1 and $g = G_2 h$. If we consider the scalar product of equation (1.24) (written for the solutions $u_n(\eta)$) by $h$, we obtain

$$\langle u_n'(\eta), h\rangle + \langle (A_1 + A_3) u_n(\eta), h\rangle + \langle BA_2 u_n(\eta), h\rangle = \langle f(\eta), h\rangle. \tag{3.8}$$

We recall that, by lemma 2, the functions $u_n(\eta)$ are $V_2$–continuous in $[0, T]$.

Integrating (3.8) between $\eta_1$ and $\eta_2 > \eta_1$, and using Hölder's inequality, it results,

$$\begin{aligned} &|\langle u_n(\eta_2), h\rangle - \langle u_n(\eta_1), h\rangle| = |(u_n(\eta_2), g)_{V_2} - (u_n(\eta_1), g)_{V_2}| \leq \\ &\leq \int_{\eta_1}^{\eta_2} \{\|A_1 u_n(\eta)\|_Z \|h\|_{Z'} + \|A_3 u_n(\eta)\|_Z \|h\|_{Z'} + \\ &+ \|BA_2 u_n(\eta)\|_{Y'} \|h\|_Y + \|f(\eta)\|_{Y'} \|h\|_Y\}\, d\eta \leq \\ &\leq \max(\|h\|_{Z'}, \|h\|_Y) \left\{ (\eta_2 - \eta_1)^{1/2} \left[ \int_{\eta_1}^{\eta_2} (\|A_1 u_n(\eta)\|_Z + \|A_3 u_n(\eta)\|_Z)^2\, d\eta \right]^{1/2} + \right. \\ &\left. + (\eta_2 - \eta_1)^{1/p} \left[ \int_{\eta_1}^{\eta_2} \|BA_2 u_n(\eta)\|_{Y'}^{p'}\, d\eta \right]^{1/p'} + (\eta_2 - \eta_1)^{1/p} \left[ \int_{\eta_1}^{\eta_2} \|f(\eta)\|_{Y'}^{p'}\, d\eta \right]^{1/p'} \right\}. \end{aligned} \tag{3.9}$$

Being, by (3.6), hypotheses IV), VI) and lemma 1,

$$\begin{aligned} &\int_0^T \|A_1 u_n(\eta)\|_Z^2\, d\eta \leq c_6 \int_0^T \|u_n(\eta)\|_W^2\, d\eta \leq K_{10}, \\ &\int_0^T \|A_3 u_n(\eta)\|_Z^2\, d\eta \leq c_7 \int_0^T \|u_n(\eta)\|_W^2\, d\eta \leq K_{11}, \\ &\int_0^T \|BA_2 u_n(\eta)\|_{Y'}^{p'}\, d\eta \leq c_8 \int_0^T \|A_2 u_n(\eta)\|_Y^p\, d\eta \leq K_{12}, \end{aligned} \tag{3.10}$$

it follows from (3.9) that the functions $(u_n(\eta), h)_H = (u_n(\eta), A_2 g)_H = (u_n(\eta), g)_{V_2}$ are uniformly Hölder-continuous $\forall g \in \{g_j\}$ (as the constants introduced in (3.10) do not depend on $n$); as we have already observed (Theorem 1), $\{g_j\}$ is a basis in $V_2$ and, consequently, the functions $u_n(\eta)$ are $V_2$–weakly uniformly continuous in $[0, T]$.

The sequence $\{(u_n(\eta), v)_{V_2}\}$ is then, $\forall v \in V_2$, uniformly bounded and uniformly continuous in $[0, T]$; by the theorem of Ascoli-Arzelà it is therefore possible ($V_2$ being separable) to extract from $\{u_n(\eta)\}$ a subsequence (again denoted by $\{u_n(\eta)\}$) such that (3.3) holds, uniformly in $[0, T]$. By (3.1), (3.2), (3.3) it is obvious that $u(\eta)$ satisfies (3.4).

LEMMA 5: *Assume that the hypotheses of theorem* 1 *hold and let* $u(\eta)$ *be the solution satisfying the initial condition*

$$u(0) = u_0 \in V_2 . \tag{3.11}$$

*It results*

$$\| u(T)\|^2_{V_2} \leq \max \{\| u_0\|^2_{V_2}, K\} \tag{3.12}$$

*where* K *is a quantity that does not depend on* $u_0$.

We start by observing that, if we apply (2.5) to the interval $[0, T]$, we obtain

$$\| u(T)\|^2_{V_2} \leq \| u_0\|^2_{V_2} + 2 \left\{ \int_0^T \| f(\eta)\|^{p'}_{Y'} d\eta \right\}^{1/p'} \left\{ \int_0^T \| A_2 u(\eta)\|^p_Y d\eta \right\}^{1/p} + \tag{3.13}$$
$$+ 2 K_1 - 2 c_3 \int_0^T \| A_2 u(\eta)\|^p_Y d\eta .$$

Assume at first that

$$\left\{ \int_0^T \| f(\eta)\|^{p'}_{Y'} d\eta \right\}^{1/p'} \left\{ \int_0^T \| A_2 u(\eta)\|^p_Y d\eta \right\}^{1/p} + K_1 \leq c_3 \int_0^T \| A_2 u(\eta)\|^p_Y d\eta . \tag{3.14}$$

Then, by (3.13),

$$\| u(T)\|^2_{V_2} \leq \| u_0\|^2_{V_2} . \tag{3.15}$$

If, instead, the relation opposite to (3.14) holds, it results, $p$ being $> 2$,

$$\int_0^T \| A_2 u(\eta) \|^p_Y d\eta \leq K_{13} . \tag{3.16}$$

There exists then, in this case, a point $\bar{\eta} \in [0, T]$ in which

$$\| u(\bar{\eta}) \|_{V_2} \leq \gamma_1 \|A_2 u(\bar{\eta})\|_H \leq \gamma_2 \| A_2 u(\bar{\eta})\|_Y \leq K_{14} , \tag{3.17}$$

$\gamma_1$ and $\gamma_2$ being embedding constants.

Applying now (2.5) to the interval $[\bar{\eta}, T]$, we obtain

$$\|u(T)\|^2_{V_2} \leq \| u(\bar{\eta})\|^2_{V_2} + 2 \left\{ \int_0^T \|f(\eta)\|^{p'}_{Y'} d\eta \right\}^{1/p'} \left\{ \int_0^T \|A_2 u(\eta)\|^p_Y d\eta \right\}^{1/p} + 2 K_1 . \tag{3.18}$$

Hence, by (3.16), (3.17),

$$\| u(\mathrm{T}) \|_{\mathrm{V}_2}^2 \leq \mathrm{K}, \tag{3.19}$$

where K is independent of $u_0$.

Relation (3.12) therefore holds in any case and the lemma is proved.

LEMMA 6: *Assume that the hypotheses of theorem 1 are verified in every bounded interval* $\subset [\eta_0, +\infty)$ *and that*

$$\operatorname*{Sup}_{t \geq \eta_0} \| f(t) \|_{\mathrm{L}^{p'}(0,1;\mathrm{Y}')} = \mathrm{M}_0 < +\infty. \tag{3.20}$$

*Then, if* $u(\eta)$ *is the solution in* $[\eta_0, +\infty)$ *satisfying the initial condition* $u(0) = u_0 \in \mathrm{V}_2$, *it results*

$$\operatorname*{Sup}_{\eta \geq \eta_0} \| u(\eta) \|_{\mathrm{V}_2} = \mathrm{M}_1' < +\infty \quad , \quad \operatorname*{Sup}_{t \geq \eta_0} \| u(t) \|_{\mathrm{L}^2(0,1;\mathrm{W})} = \mathrm{M}_2' < +\infty, \tag{3.21}$$

$$\operatorname*{Sup}_{t \geq \eta_0} \| \mathrm{A}_2 u(t) \|_{\mathrm{L}^{p}(0,1;\mathrm{Y})} = \mathrm{M}_3' < +\infty \; , \; \operatorname*{Sup}_{t \geq \eta_0} \| \mathrm{A}_2 u(t) \|_{\mathrm{H}^{\varepsilon}(0,1;\mathrm{D}(\mathrm{A}_1^{\varepsilon}))} = \mathrm{M}_4' < +\infty, \tag{3.22}$$

*the quantities* $\mathrm{M}_j'$ *being independent of* $\eta_0$ *and* $\varepsilon$ *being an appropriate positive number.*

Let us divide the half-axis $\eta \geq \eta_0$ in segments of unitary length by means of the points $\eta_j = \eta_0 + j$ $(j = 1, 2, \cdots)$ and denote by $\eta_j'$ the point of the interval $[\eta_{j-1}, \eta_j]$ in which the continuous function $\| u(\eta) \|_{\mathrm{V}_2}$ takes on its maximum value.

Let us further set

$$\mathrm{M} = \max_{\eta_0 \leq \eta \leq \eta_2} \| u(\eta) \|_{\mathrm{V}_2}. \tag{3.23}$$

Assume that

$$\| u(\eta_3') \|_{\mathrm{V}_2} > \| u(\eta_1') \|_{\mathrm{V}_2}. \tag{3.24}$$

It is then, applying (2.5) to the interval $[\eta_1', \eta_3']$ and bearing in mind that $1 \leq \eta_3' - \eta_1' \leq 3$,

$$\begin{aligned} \| u(\eta_1') \|_{\mathrm{V}_2}^2 &< \| u(\eta_3') \|_{\mathrm{V}_2}^2 \leq \| u(\eta_1') \|_{\mathrm{V}_2}^2 + 2 \int_{\eta_1'}^{\eta_3'} \| f(\eta) \|_{\mathrm{Y}'} \| \mathrm{A}_2 u(\eta) \|_{\mathrm{Y}} \, d\eta + \\ &+ 2 \mathrm{K}_1 - 2 c_3 \int_{\eta_1'}^{\eta_3'} \| \mathrm{A}_2 u(\eta) \|_{\mathrm{Y}}^{p} \, d\eta \leq \\ &\leq \| u(\eta_1') \|_{\mathrm{V}_2}^2 + 2 \cdot 3^{1/p'} \mathrm{M}_0 \left\{ \int_{\eta_1'}^{\eta_3'} \| \mathrm{A}_2 u(\eta) \|_{\mathrm{Y}}^{p} \, d\eta \right\}^{1/p} + 2 \mathrm{K}_1 - 2 c_3 \int_{\eta_1'}^{\eta_3'} \| \mathrm{A}_2 u(\eta) \|_{\mathrm{Y}}^{p} \, d\eta . \end{aligned} \tag{3.25}$$

Hence

$$(3.26) \qquad c_3 \int_{\eta_1'}^{\eta_3'} \| A_2 u(\eta) \|_Y^p \, d\eta \leq 3^{1/p'} M_0 \left\{ \int_{\eta_1'}^{\eta_3'} \| A_2 u(\eta) \|_Y^p \, d\eta \right\}^{1/p} + K_1$$

and, consequently,

$$(3.27) \qquad \int_{\eta_1'}^{\eta_3'} \| A_2 u(\eta) \|_Y^p \, d\eta \leq K_{15} .$$

There exists therefore a point $\eta^* \in [\eta_1', \eta_3']$ in which

$$(3.28) \qquad \| u(\eta^*) \|_{V_2} \leq \gamma_1 \| A_2 u(\eta^*) \|_H \leq \gamma_2 \| A_2 u(\eta^*) \|_Y \leq K_{16} .$$

Applying again (2.5) to the interval $[\eta^*, \eta_3']$, we then obtain, by (3.21), (3.27), (3.28),

$$(3.29) \qquad \| u(\eta_3') \|_{V_2}^2 \leq \| u(\eta^*) \|_{V_2}^2 + 2 \cdot 3^{1/p'} M_0 \left\{ \int_{\eta^*}^{\eta_3'} \| A_2 u(\eta) \|_Y^p \, d\eta \right\}^{1/p} + 2 K_1 \leq K_{17} .$$

If, on the other hand, (4.5) does not hold, it results

$$(3.30) \qquad \| u(\eta_3') \|_{V_2} \leq \| u(\eta_1') \|_{V_2} .$$

Hence, by (3.29), (3.30), it is, in any case,

$$(3.31) \qquad \| u(\eta_3') \|_{V_2} \leq \max (\| u(\eta_1') \|_{V_2} , K_{17}) \leq \max (M , K_{17}) ,$$

where $K_{17}$ and M do not depend on $\eta_0$.

Repeating the procedure for the points $\eta_4', \eta_5', \cdots$ we obtain, in a similar way, that

$$(3.32) \qquad \| u(\eta_j') \|_{V_2} \leq \max (M , K_{17}) \qquad (j = 1, 2, \cdots)$$

and the first of relations (3.21) is proved.

The second of (3.21) and the first of (3.22) can be obtained directly by applying (2.5) to any interval $[\eta, \eta + 1] \subset [\eta_0, +\infty)$ and bearing in mind (3.32). Finally, the second of (3.22) follows from (3.21), the first of (3.22) and Lemma 3.