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Structure Theory in s-d-Rings. Nota II

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Matematica. — *Structure Theory in s–d–Rings*. Nota II di ESAYAS GEORGE KUNDERT, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Proseguendo lo studio iniziato in una Nota precedente [1], si generalizza la nozione ivi introdotta di *s–d*–anello, allo scopo di stabilire opportuni legami di tali generalizzazioni con certe algebre di Hopf (teor. 1). Si mostra poi (teor. 2) come le *s–d*–anelli possano venire caratterizzate nettamente entro la totalità degli *s–d*–anelli generalizzati.

1. Definition: A generalized *d*–ring is a ring $\mathfrak{A}$ defined just like a *d*–ring (see [1], paragraph 1) with the exception that we do not demand condition 1 (namely that d is an onto mapping), and we replace condition 3 by:

$$3' \qquad d(ab) = \Sigma \varepsilon_{ij} d^{(i)} a d^{(j)} b \quad , \quad \varepsilon_{ij} \in \mathrm{R} , \qquad i , j \geq 0 .$$

By R we mean again the ring of constants. (We do not assume that $\varepsilon_{ij} = 0$ for large i , j). Using $a = b = 1 \Rightarrow \varepsilon_{00} = 0$. In the following we consider $\mathfrak{A}$ again (in the natural way) as an R–algebra.

Definition: A *d*–basis in a generalized *d*–ring is a (non finite) countable free basis $\{x_0 = 1 , x_1 , \cdots , x_n , \cdots\}$ such that $dx_n = x_{n-1}$ for all $n \geq 1$.

PROPOSITION 1: *The existence of a d–basis is equivalent to condition* 1 ([1], *paragraph* 1), *namely that d is an onto mapping.*

Proof: If we have a *d*–basis and $a = \sum_{i=0}^{n} \alpha_i x_i \in \mathfrak{A}$. Take $a' = \sum_{i=0}^{n} \alpha_i x_{i+1}$ then $da' = a \Rightarrow d$ is onto. If d is onto, let $x_1 \in \mathfrak{A}$ such that $dx_1 = 1$, x_i such that $dx_i = x_{i-1}$. It is clear that $x_n \neq x_i$ for all $i < n$ because $x_n = x_i \Rightarrow \Rightarrow d^{(n)} x_n = 1 = d^{(n)} x_i = 0$.

$\{1 , x_1 , x_2 , \cdots , x_n , \cdots\}$ is therefore an infinite countable sequence of elements of $\mathfrak{A}$. That this forms a free basis for the R–algebra $\mathfrak{A}$ is proved exactly like the proposition in paragraph 2 [1].

2. Definition: A generalized *s–d*–ring is a generalized *d*–ring satisfying condition 1, and having a σ homomorphism as defined in [1], paragraph 2 (i.e., an *s–d*–ring with condition 3′ instead of 3).

Definition: A σ–*d*–basis is a *d*–basis such that the mapping $\sigma(a) = \alpha_0$ where $a = \sum_{i=0}^{n} \alpha_i x_i$ is a ring-homorphism.

PROPOSITION 2: *The existence of a σ–d–basis is a necessary and sufficient condition for a generalized d–ring to be a generalized s–d–ring.*

(*) Nella seduta del 14 novembre 1967.

Proof: If we have a σ–d–basis $\Rightarrow$ condition 1 holds by proposition 1, and our σ is a ring-homomorphism such that $\sigma(a) = a$ for $a \in R$. If we give a generalized s–d–ring we may construct an integration s and then a σ–d–basis exactly as it was done in [1], paragraph 2.

A G–Hopf algebra $\mathfrak{A}$ over a ring R is a (non graded) commutative Hopf algebra with an infinitely countable basis $\{1, x_1, x_2, \cdots, x_n, \cdots\}$ such that the co-multiplication $\Delta : \mathfrak{A} \rightarrow \mathfrak{A} \otimes \mathfrak{A}$ is as follows: $\Delta(x_n) = \sum_{i+j=n} x_i \otimes x_j$.

THEOREM I: *The ring structure belonging to a generalized s–d–ring can be enriched to a G–Hopf algebra structure and the ring structure belonging to a G–Hopf algebra can be enriched to a generalized s–d–ring structure.*

Proof: 1. Let $\mathfrak{A}$ be a s–d–ring. By Proposition 2, there exists a σ–d–basis $\{1, x_1, \cdots, x_n, \cdots\}$ for the natural R–algebra belonging to $\mathfrak{A}$. Let $x_i \cdot x_j = \sum_{s=0}^{n} \alpha_{ij}^{s} x_s$. Since we assume commutativity we may assume $i \leq j$. Taking $i = 0$ we get

$$(1) \qquad \alpha_{0i}^{s} = 0 \quad , \quad i \neq s \quad , \quad \alpha_{0s}^{s} = 1 .$$

From $\sigma(x_i) = 0$ if $i \geq 1 \Rightarrow \sigma(x_i x_j) = 0 \Rightarrow$

$$(2) \qquad \Rightarrow \alpha_{ij}^{0} = 0 \quad \text{if not} \quad i = j = 0.$$

From $d(x_i x_j) = \sum_{s=1}^{n} \alpha_{ij}^{s} x_{s-1} = \sum_{r,t} \varepsilon_{rt}\, x_{i-r} x_{j-t} =$

$$= \sum_{s=1}^{n} \Big(\sum_{r,t} \varepsilon_{rt}\, \alpha_{i-r,j-t}^{s-1} \Big) x_{s-1} \Rightarrow \alpha_{ij}^{s} = \sum_{r,t} \varepsilon_{rt}\, \alpha_{i-r,j-t}^{s-1} \Rightarrow$$

$$(3) \qquad \Rightarrow \alpha_{ij}^{s} = \sum_{\substack{i_1 + \cdots + i_s = i \\ j_1 + \cdots + j_s = j}} \varepsilon_{i_1 j_1}\, \varepsilon_{i_2 j_2} \cdots \varepsilon_{i_s j_s} \Rightarrow$$

$$(4) \qquad \Rightarrow \alpha_{ij}^{r+t} = \sum_{\substack{h+g=i \\ l+k=j}} \alpha_{hl}^{r}\, \alpha_{gk}^{t} .$$

$\Big($From (3) we also get: $\alpha_{ij}^{1} = \varepsilon_{ij}$, $\alpha_{ij}^{i+j} = \binom{i+j}{i}$, $\alpha_{ij}^{s} = 0$ for all $s > i + j\Big)$.

To construct now our G–Hopf algebra, we take our R–algebra and as basis our σ–d–basis $\{1, x_1, \cdots, x_n, \cdots\}$. We define the co-unit $\varepsilon : \mathfrak{A} \rightarrow R$ by letting $\varepsilon = \sigma$, and we must define the co-multiplication $\Delta : \mathfrak{A} \rightarrow \mathfrak{A} \otimes \mathfrak{A}$ by letting $\Delta x_n = \sum_{r+t=n} x_r \otimes x_t$ and then $\Delta a = \Delta \sum_{i=0}^{n} \alpha_i x_i = \sum_{i=0}^{n} \alpha_i \Delta x_i$. We have to show that Δ is an algebra homomorphism. It will be enough to show: $\Delta(x_i x_j) = \Delta x_i \cdot \Delta x_j$, but $\Delta(x_i x_j) = \sum_{s} \alpha_{ij}^{s} \Delta x_s = \sum_{\substack{s \\ r+t=s}} \alpha_{ij}^{s} (x_r \otimes x_t)$ and this is by (4)

$$= \sum_{\substack{h+g=i \\ l+k=j}} \alpha_{hl}^{r}\, \alpha_{gk}^{t} (x_r \otimes x_t) = \sum_{\substack{h+g=i \\ l+k=j}} \Big(\sum_{r} \alpha_{hl}^{r} x_s \Big) \otimes \Big(\sum_{t} \alpha_{gk}^{t} x_t \Big) = \sum_{\substack{h+g=i \\ l+k=j}} x_h x_l \otimes x_g x_k = \Delta x_i \cdot \Delta x_j .$$

Let $\varphi_1 : R \otimes \mathfrak{A} \rightarrow \mathfrak{A}$, $\varphi_2 : \mathfrak{A} \otimes R \rightarrow \mathfrak{A}$ be the canonical isomorphisms, then the conditions on the co-unit, namely $\varphi_2 \circ (\varepsilon \otimes 1) \circ \Delta = \text{id.}$ and $\varphi_1 \circ (1 \otimes \varepsilon) \circ \Delta = \text{id.}$ are clearly satisfied.

2. Let now $\mathfrak{A}$ be a G–Hopf algebra, $\{1, x_1, \cdots, x_n, \cdots\}$ a basis with the property stated in the definition of a G–Hopf algebra. Let $a = \sum_{i=0}^{n} \alpha_i x_i \in \mathfrak{A}$. Define: $da = \sum_{i=0}^{n} \alpha_i x_{i-1}$ (where $x_{-1} = 0$ by definition). Conditions 2, 4, 5, for a generalized d–ring are clearly satisfied. To prove condition 3′ it will be enough to prove it for $a = x_i$, $b = x_j$, $i \leq j$. Since Δ is a homomorphism we have: $\Delta(x_i x_j) = \Sigma \overset{s}{\alpha}_{ij} \Delta x_s = \sum_{r,t} \overset{r+t}{\alpha}_{ij} (x_r \otimes x_t)$ *is* $= \Delta x_i \cdot \Delta x_j = \Big(\sum_{l+k=i} x_l \otimes x_k\Big)\Big(\sum_{g+h=j} x_g \otimes x_h\Big) =$
$= \sum_{\substack{l+k=i \\ g+h=j}} x_l x_g \otimes x_k x_h = \sum_{\substack{r,t \\ l+k=i \\ g+h=j}} \overset{r}{\alpha}_{lg} \overset{t}{\alpha}_{kh} x_r \otimes x_t \Rightarrow \overset{r+t}{\alpha}_{ij} = \sum_{\substack{l+k=i \\ g+h=j}} \overset{r}{\alpha}_{lg} \overset{t}{\alpha}_{kh}$, and specially

$$\overset{s}{\alpha}_{ij} = \sum_{r,t} \overset{1}{\alpha}_{rt} \cdot \overset{s-1}{\alpha}_{i-r\, j-t} \,. \tag{6}$$

Define now: $\varepsilon_{ij} = \overset{1}{\alpha}_{ij}$. Then $d(x_i x_j) = \sum_s \overset{s}{\alpha}_{ij} x_{s-1} = \sum_s \Big(\sum_{r,t} \overset{1}{\alpha}_{rt} \overset{s-1}{\alpha}_{i-r\, j-t}\Big) x_{s-1} =$
$= \sum_{r,t} \varepsilon_{rt} \Big(\sum_s \overset{s-1}{\alpha}_{i-r j-t} x_{s-1}\Big) = \sum_{r,t} \varepsilon_{rt} x_{i-r} x_{j-t} = \Sigma \varepsilon_{rt} d^{(r)} x_i \cdot d^{(t)} x_j$ which is condition 3′.

REMARKS. (*a*) We could have formulated Theorem 1 differently by introducing the dual concept of a s'–d'–co–ring and then show that there is a 1–1–correspondence between s–d–rings and s'–d'–co–rings and that corresponding rings can be combined into a G–Hopf algebra. That gives us an intrinsic definition of a G–Hopf algebra and the proof can also be formulated without using a basis. This will be done in a final version of the theory.

(*b*) An obvious question is now the following: Given a free R–module with countable basis, what are the possible generalized s–d–rings (or G–Hopf algebras) over R? It will be enough to define the products $x_i x_j$ for a given basis $\{1, x_1, \cdots, x_n, \cdots\}$. From our proof above we know that we have to put $\varepsilon_{ij} = \overset{1}{\alpha}_{ij}$ and that all $\overset{s}{\alpha}_{ij}$ are determined by the $\overset{1}{\alpha}_{ij}$. The associative law however demands further relations between the ε_{ij} and it depends on the nature of the ring R how generally these can be satisfied. If for example the field of rational numbers is contained in R, then we can freely choose the ε_{1i} and all the other ε_{ij} are well determined. If we choose ε_{11}, ε_{12} arbitrarily but $\varepsilon_{1i} = 0$ for $i > 2$ then we get possible generalized s–d–rings, no matter what R is. Had we demanded in 3′ that $\varepsilon_{ij} = 0$ for large i, j, then the only generalized s–d–rings would be those with $\varepsilon_{ij} = 0$ for $i, j > 1$ if R is an integral domain, but this condition does not seem to be a natural one, at least not in this context. A student of mine is now investigating this question more carefully.

4. To single out our (special) s–d–ring as defined in [1], paragraph 1, we define the notion of τ–basis.

D e f i n i t i o n: A τ–basis in a generalized d–ring is a basis such that the mapping τ: $\mathfrak{A} \rightarrow$ R defined by $\tau\Big(\sum_{i=0}^{n} \alpha_i x_i\Big) = \sum_{i=0}^{n} \alpha_i$ is a ring homomorphism. Examples of generalized d–ring with τ–basis are:

(a) Our s–d–ring from [1].

(b) The polynomial ring R [X] over a ring R, with the formal derivation, has a τ–basis, namely: $x_i = X^i$, $i = 0, 1, 2, \cdots$. Observe that this is not a d–basis. If R contains the rationals, then $x_i = \frac{X^i}{i!}$ is a σ–d–basis but this is not a τ–basis. A polynomial ring can never be a generalized s–d–ring with a s–d–τ–basis. That follows as a corollary to the following theorem.

THEOREM II: *The only generalized s–d–rings which have a s–d–τ–basis are the s–d–rings.*

Proof: $\tau(x_i x_1) = 1 = \overset{2}{\alpha}_{11} + \overset{1}{\alpha}_{11}$, but $\overset{2}{\alpha}_{11} = 2 \Rightarrow \overset{1}{\alpha}_{11} = -1$. $\tau(x_i x_2) = = \overset{3}{\alpha}_{12} + \overset{2}{\alpha}_{12} + \overset{1}{\alpha}_{12} = 1$, since $\overset{3}{\alpha}_{12} = 3$, $\overset{2}{\alpha}_{12} = 2\overset{1}{\alpha}_{11} = -2 \Rightarrow \overset{1}{\alpha}_{12} = 0$. Suppose now that we have already proved that $\overset{1}{\alpha}_{st} = 0$ for $(s, t) \neq (0, 1)$, $(1, 1)$ and $s < i$, $t < j$ assuming $i, j \geq 2$. Since by (4) $\overset{s}{\alpha}_{ij} = \sum_{\substack{h+g=i \\ l+k=j}} \overset{1}{\alpha}_{hl} \overset{s-1}{\alpha}_{gk} = \overset{s-1}{\alpha}_{ij-1} + + \overset{s-1}{\alpha}_{i-1j} - \overset{s-1}{\alpha}_{i-1j-1}$ for $s \geq 2$ and $\tau(x_i x_j) = 1 = \overset{1}{\alpha}_{ij} + \sum_{s=2}^{i+j} \overset{s}{\alpha}_{ij} = \overset{1}{\alpha}_{ij} + \sum_{s=2}^{i+j} \overset{s-1}{\alpha}_{ij-1} + + \sum_{s=2}^{i+j} \overset{s-1}{\alpha}_{i-1j} - \sum_{s=2}^{i+j} \alpha_{i-1j-1} = \overset{1}{\alpha}_{ij} + \tau(x_i x_{j-1}) + \tau(x_{i-1} x_j) - \tau(x_{i-1} x_{j-1}) = \overset{1}{\alpha}_{ij} + + 1 + 1 - 1 \Rightarrow \overset{1}{\alpha}_{ij} = 0$ for $(i, j) \neq (0, 1), (1, 0), (1, 1)$. But $\overset{1}{\alpha}_{ij} = \varepsilon_{ij} \Rightarrow d(ab) = = adb + bda - dadb$, which is condition 3 in paragraph 1 [1].

The simultaneous existence of the σ– and τ–homomorphism in a s–d–ring allows us to partially study the multiplicative structure of inteals in s–d–rings. This will be done in Note III.

LITERATURE.

[1] E. G. KUNDERT, *Structure Theory in s–d–Rings*, Nota I, « Rend. Acc. Naz. Lincei », Ser. VIII, vol. XLI, fasc. 5, november 1966, pp. 270–278.