Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

# Rendiconti

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## Double transitivity in finite affine and projective planes

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8*, Vol. **43** (1967), n.5, p. 317–320.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1967_8_43_5_317_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1967.

**Matematica.** — *Double transitivity in finite affine and projective planes*. Nota di JUDITA COFMAN, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Ostrom e Wagner [11] hanno dimostrato che, se un piano affine (proiettivo) finito ammette un gruppo di collineazioni 2-transitivo sui punti, allora il piano è un piano di traslazione (desarguesiano).

Siano $\mathfrak{P}^*$ ($\mathfrak{P}$) un piano affine (proiettivo) finito ed $\mathcal{O}$ un suo sottoinsieme di punti, tali che vi sia un gruppo di collineazioni del piano che trasformi $\mathcal{O}$ in se e che sia 2-transitivo sui punti di $\mathcal{O}$; allora è possibile dimostrare, sotto opportune ipotesi addizionali, che i punti di $\mathcal{O}$ costituiscono un sottopiano di $\mathfrak{P}^*$ ($\mathfrak{P}$). Questa Nota riassume i risultati ottenuti sulla questione; per le dimostrazioni si rinvia alla Bibliografia qui data alla fine.

## 1. – INTRODUCTION.

Investigating projective and affine planes with transitive collineation groups Ostrom and Wagner [11] have proved the following results:

THEOREM A: *Let $\mathfrak{P}^*$ be a finite affine plane and $\Delta$ be a collineation group of $\mathfrak{P}^*$ doubly transitive on the affine points of $\mathfrak{P}^*$. Then $\mathfrak{P}^*$ is a translation plane and $\Delta$ contains the translation group of $\mathfrak{P}^*$ as a subgroup.*

THEOREM B: *Let $\mathfrak{P}$ be a finite projective plane and let $\Delta$ be a collineation group of $\mathfrak{P}$ doubly transitive on the points of $\mathfrak{P}$. Then $\mathfrak{P}$ is desarguesian and $\Delta$ contains the little projective group of $\mathfrak{P}$.*

It is natural to state the following question:

Let $\mathfrak{P}^*$ ($\mathfrak{P}$) be a finite (projective) plane with a subset $\mathcal{O}$ of points admitting a collineation group $\Delta$ which maps $\mathcal{O}$ onto itself and induces a doubly transitive permutation group on the points of $\mathcal{O}$. What can we say about the plane, the set $\mathcal{O}$ and the collineation group $\Delta$?

The purpose of this note is to give a catalogue of results which I have obtained concerning the above problem. The proofs of these results will be published elsewhere. While the methods applied by Ostrom and Wagner in [11] are elementary, my approach is based on deep group-theoretical statements.

## 2. MAIN RESULTS.

For definitions of an affine and projective plane, translation plane and desarguesian plane, collineations, perspectivities, of a translation group and little projective group see for instance Pickert [12].

An involution of an affine or projective plane is a collineation of order two.

(*) Nella seduta del 14 novembre 1967.

Concerning affine planes I have been able to prove the following statement:

THEOREM I: *Let* $\mathcal{S}^*$ *be a finite affine plane of order* $n$ *containing a subset* $\mathcal{O}$ *of* $k$ *affine points. Let* $\Delta$ *be a collineation group of* $\mathcal{S}^*$ *mapping* $\mathcal{O}$ *onto itself and inducing a doubly transitive permutation group on the poins of* $\mathcal{O}$. *If the involutions of* $\Delta$ *are perspective then*

(*a*) (see Cofman [2]):

*if* $k > n + 1$ *then* $\mathcal{O}$ *consists of all affine points of* $\mathcal{S}^*$, *the plane* $\mathcal{S}^*$ *is a translation plane and* $\Delta$ *contains the translation group of* $\mathcal{S}^*$:

(*b*) (see Cofman [3]):

*if*

(i) $n$ *is even*,
(ii) $2 < k \leq n + 1$,
(iii) $\Delta$ *is non-soluble*,
(iv) *the points of* $\mathcal{O}$ *are not all collinear but at least three of them are collinear*,

*then the points of* $\mathcal{O}$ *form a proper affine subplane* $\mathcal{S}_0^*$ *of* $\mathcal{S}^*$ *and* $\Delta$, *restricted to* $\mathcal{S}_0^*$, *contains the translation group of* $\mathcal{S}_0^*$:

(*c*) (see Cofman [4]):

*if*

(i) $n$ *is odd*,
(ii) $2 < k \leq n + 1$,
(iii) $\Delta$ *is non-soluble*,
(iv) *the points of* $\mathcal{O}$ *are not all collinear but at least three of them are collinear*,
(v) *non non-identical collineation of* $\Delta$ *fixes a proper subplane of* $\mathcal{S}^*$,

*then the points of* $\mathcal{O}$ *form a proper affine subplane* $\mathcal{S}_0^*$ *of* $\mathcal{S}^*$ *and* $\Delta$, *restricted to* $\mathcal{S}_0^*$ *contains, the translation group of* $\mathcal{S}_0^*$.

The assumption that the involutions of $\Delta$ are perspectivities is probably superfluous. In both cases (*b*) and (*c*) restriction (iii) cannot be eliminated because, as T. G. Ostrom has pointed out to me, the finite translation planes $\mathcal{A}$ of André [1] of order $n$ admit soluble collineation groups acting doubly transitively on the elements of a set $\mathcal{O}$ of $n$ affine points such that the points of $\mathcal{O}$ satisfy condition (iv) but do not form a subplane of $\mathcal{A}$. Restriction (iv) is also essential since there are: (1) examples of finite affine planes with collineation groups $\Delta$ acting doubly transitively on the points of an affine line fixed under $\Delta$ (for instance finite desarguesian planes or the Ostrom–Rosati planes (see Ostrom [9])) and (2) examples of finite desarguesian affine planes admitting collineation groups which fix a set $\mathcal{O}$ of affine points no three of which are collinear inducing a doubly transitive permutation group on the points of $\mathcal{O}$. (I could not find examples of non-desarguesian planes with this last property).

The investigation of finite projective planes presents more difficulties. A similar result to theorem I can be obtained if instead of double transitivity of $\Delta$ on a subset $\mathcal{O}$ of points in the plane we assume that $\Delta$ is transitive on the " ordered triangles " of $\mathcal{O}$:

THEOREM II (Cofman [5]: *Let $\mathcal{P}$ be a finite projective plane of order $n$ with a subset $\mathcal{O}$ of $k$ points such that the points of $\mathcal{O}$ are not all collinear but at least three of them are collinear. Let $\Delta$ be a collineation group of $\mathcal{P}$ which maps $\mathcal{O}$ onto itself and is transitive on the ordered non-collinear triplets of points of $\mathcal{O}$. If the involutions of $\Delta$ are perspectivities then either the points of $\mathcal{O}$ form an affine subplane $\mathcal{P}_0^*$ of $\mathcal{P}$ and $\Delta$, restricted to $\mathcal{P}_0^*$, contains the translation group of $\mathcal{P}_0^*$, or the points of $\mathcal{O}$ form a desarguesian subplane $\mathcal{P}$ of $\mathcal{P}_0$ and $\Delta$, restricted to $\mathcal{P}_0$, contains the little projective group of $\mathcal{P}_0$.*

In the case when $\Delta$ is doubly transitive on the elements of a subset of points in a finite projective plane I could prove the following two theorems:

THEOREM III (unpublished): *Let $\mathcal{P}$ be a finite projective plane of order $n$ with a subset $\mathcal{O}$ of $k > (n^2 + n)/2$ points. Let $\Delta$ be a collineation group of $\mathcal{P}$ fixing $\mathcal{O}$ and acting doubly transitively on the elements of $\mathcal{O}$. If the involutions of $\Delta$ are perspectivities then either $\mathcal{O}$ consists of all points of $\mathcal{P}$, the plane is desarguesian and $\Delta$ contains the little projective group of $\mathcal{P}$ or $\mathcal{O}$ consists of $n^2$ elements which form an affine subplane $\mathcal{P}_0^*$ of $\mathcal{P}$; the plane $\mathcal{P}_0^*$ is a translation plane and $\Delta$ contains the translation group of $\mathcal{P}_0^*$.*

THEOREM IV: *Let $\mathcal{P}$ be a finite projective plane of odd order $n \not\equiv 1 \pmod 8$ or of prime order $n$ and let $\mathcal{O}$ be a set of $n+1$ points of $\mathcal{P}$. If $\mathcal{P}$ admits a collineation group fixing $\mathcal{O}$ and acting doubly transitively on the points of $\mathcal{O}$ then either*

(a) (see Cofman [6]) *the points of $\mathcal{O}$ are collinear, $\mathcal{P}$ is desarguesian and $\Delta$ contains the special linear group* SL $(2, n)$, *or*

(b) (unpublished) *no three points of $\mathcal{O}$ are collinear, $\mathcal{P}$ is desarguesian and $\Delta$ contains the projective special linear group* PSL $(2, n)$ (1).

Projective planes of even order admitting a collineation group $\Delta$ which acts doubly transitively on the points of a line are not all desarguesian. This is illustrated by the example of the Tits-Lüneburg planes (see Tits [13] and Lüneburg [8]).

The above investigations raise the following question: *Do there exist planes of order $n$ satisfying the conditions of Theorems I and II for $k < n^2$?*

The answer is affirmative since finite desarguesian planes have the required properties. Moreover the affine Hughes planes and the projective Hughes planes of order $n$ are examples of strict semi-translation planes (see Ostrom [10]) which satisfy the conditions of Theorems I–II for $k = n$ and $k = n + \sqrt{n} + 1$ respectively.

(1) For the definitions of SL $(2, n)$ and PSL $(2, n)$ see Dickson [7].

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