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On some subspaces for operators of class(N)I

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Analisi matematica. — *On some subspaces for operators of class* (N) I. Nota di VASILE ISTRĂŢESCU, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Un operatore T appartiene alla classe (N) se per ogni elemento x, $\|x\| = 1$, $\| Tx \|^2 \leq \| T^2 x \|$. Allora $\mathfrak{I}(T) = \{x, \| T^n x \| \leq \| x \|, n = 1, 2, \cdots\}$ è un sotto-spazio chiuso e invariante per T e per ogni operatore e commuta con T.

Nel caso degli operatori normali questo risultato è stato dimostrato da P. R. Halmos.

1. If A is a normal operator on a Hilbert space H, the subspace

$$\mathfrak{I}(A) = \{x, x \in H, \|A^n x\| \leq \|x\|\}$$

is a closed invariant subspace for A [1]. The aim of this Note is to obtain the same result for more general classes of operators, namely hyponormal operators and operators of class (N) [2], [3].

2. An operator T on a Hilbert space H is of class (N) if

$$\| Tx \|^2 \leq \| T^2 x \|$$

for every element $x \in H$, $\| x \| = 1$. It is easy to see that every hyponormal operator (An operator T is hyponormal if $\| T^* x \| \leq \| Tx \|$ for every element $x \in H$) is of class (N). The following result is known and we give it here for the completion.

THEOREM 2.1. *If* T *is of class* (N) *then* T^n *is also of class* (N) *for all integers* n.

Proof. We have, from the definition of the class (N) that

$$\|T\| \geq \cdots \geq \frac{\|T^{n+1} x\|}{\|T^n x\|} \geq \cdots \geq \frac{\|T^3 x\|}{\|T^2 x\|} \geq \frac{\|T^2 x\|}{\|Tx\|} \geq \frac{\|Tx\|}{\|x\|}$$

which implies, obviously, that

$$\| T^{2n} x \| \geq \| T^n x \|^2$$

for every unit element x in H and the theorem is proved.

THEOREM 2.2. *If* T *is of class* (N) *and if* $\mathfrak{I} = \mathfrak{I}(T)$ *is the set of all elements* x *such that* $\| T x^n \| \leq \| x \|$ *for every positive integer* n *then* $\mathfrak{I}$ *is a subspace. If* B *is an operator which commutes with* T *then* $\mathfrak{I}$ *is invariant under* B.

Remark. For the case when T is normal this was proved by P. R. Halmos [1].

(*) Nella seduta del 14 novembre 1967.

Proof. Let m be the set of all elements n such that the sequence $\{\|T^n x\|\}$ is bounded. If x and y are in m and α and β are complex numbers, then the relation

$$\|T^n(\alpha x + \beta y)\| \leq |\alpha| \, \|T^n x\| + |\beta| \, \|T^n y\|$$

valid for every positive integer n shows that $\alpha x + \beta y \in m$. Also, if $x \in m$ then the relation $\|T^n Bx\| = \|BT^n x\| \leq \|B\| \, \|T^n x\|$ shows that $Bx \in m$. In other words m is a linear manifold, and is invariant under B. It is clear that $\mathfrak{J} \subset m$. It is not clear that m is closed. For this, we show that $\mathfrak{J} = m$. It is sufficient to show that if an element x is such that $\|T^m x\| > \alpha$ $(\|x\| = 1)$ for some integer m and for some $\alpha > 1$ then $\{\|T^n x\|\}$ is not bounded. But this follows from the fact that T^m is of class (N) by theorem 2.1.

The theorem is proved.

Consider, now that T is a hyponormal operator. It is known that for all complex numbers λ, μ the operator $\lambda T + \mu$ is also hyponormal. Let λ be a complex number and ε be a positive real number; let be the hyponormal operator $\frac{1}{\varepsilon}(T - \lambda)$ and $\mathfrak{J}\left(\frac{1}{\varepsilon}(T - \lambda)\right) = I(\lambda, \varepsilon)$. It is clear that an element x is in $I(\lambda, \varepsilon)$ if

$$\|(T - \lambda)^n x\| \leq \varepsilon^n \|x\|.$$

Also, the subspace $I(\lambda, \varepsilon)$ is closed and thus the spectrum of $(T - \lambda)/\mathfrak{J}(\lambda, \varepsilon)$ is contained in the circle $\{z, |z| \leq \varepsilon\}$. This leads to the fact that the spectrum $T/\mathfrak{J}(\lambda, \varepsilon)$ is contained in $\{z, |z - \lambda| \leq \varepsilon\}$.

Remark 1. The elegant and simple proof of theorem 2.1 given here is communicated to the author by T. Furuta ([4] Addendum).

Remark 2. The theorems 2.1 and 2.2 are also valid for operators on Banach spaces.

References.

[1] P. Halmos, *Introduction to Hilbert space*, Chelsea PC. N.Y. 1951.
[2] S. Berberian, *A note on hyponormal operators*, « Pacif. J. Math. », *12*, 4 (1962).
[3] V. Istrățescu, *On some hyponormal operators*, « Pacif. J. Math. », *22*, 2 (1967).
[4] T. Furuta, *On the class of Paranormal operators* (to appear).