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Giovanni Prouse

Periodic or almost-periodic solutions of a non linear functional equation. Nota II

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Analisi matematica. — *Periodic or almost-periodic solutions of a non linear functional equation.* Nota II (*) di GIOVANNI PROUSE, presentata dal Corrisp. L. AMERIO.

SUNTO. – Dopo aver dimostrato alcuni lemmi preliminari, si dà la dimostrazione del teorema 1 enunciato nel § 1.

2. Let us at first prove the following three fundamental lemmas.

LEMMA 1: *Assume that conditions* I) *and* II) *hold and that* $v \in W$. *Then* $A_1 v \in Z$, $A_2 v \in Z'$.

We recall that the operators A_i $(i = 1, 2)$ are linear and continuous from $D(A_i)$ on H and that the domains $D(A_i)$ are dense in H; as the operators A_i are self adjoint, it results, by transposition, that they are also linear and continuous from H on $D(A_i)'$. Moreover, $D(A_1 A_2)$ is dense in H, being, by hypothesis I), $D(A_1 A_2) \supset D(A_1^2)$.

Consequently (Lions [1]),

$$W = D((A_1 A_2)^{1/2}) = [D(A_1 A_2), H]_{1/2} = [D(A_2 A_1), H]_{1/2}. \tag{2.1}$$

It is also, bearing in mind (1.15), (1.16),

$$A_1 D(A_2 A_1) = A_1 G_1 G_2 H = G_2 H = D(A_2), \tag{2.2}$$

$$A_2 D(A_1 A_2) = A_2 G_2 G_1 H = G_1 H = D(A_1). \tag{2.3}$$

From relations (2.1), (2.2), (2.3) we obtain, by an interpolation theorem of Lions [2] and a duality theorem also of Lions [3],

$$\begin{aligned} &A_1 v \in [A_1 D(A_1 A_2), A_1 H]_{1/2} = [A_1 D(A_2 A_1), A_1 H]_{1/2} = [D(A_2), D(A_1)']_{1/2} = Z, \\ &A_2 v \in [A_2 D(A_1 A_2), A_2 H]_{1/2} = [D(A_1), D(A_2)']_{1/2} = Z' \end{aligned} \tag{2.4}$$

and the lemma is proved.

LEMMA 2: *Assume that hypotheses* I), II), III), IV), VI), VIII) *hold and let* $u(\eta)$ *be a solution in* $[0, T]$ *of* (1.24), *with* $f \in L^{p'}(0, T; Y')$. *The function* $u(\eta)$ *is then* V_2*-continuous in* $[0, T]$ *and* $\forall$ *two points* η_1 *and* $\eta_2 > \eta_1$ *of* $[0, T]$, *the following "energy" relation holds*

$$\frac{1}{2}\|u(\eta_2)\|_{V_2}^2 - \frac{1}{2}\|u(\eta_1)\|_{V_2}^2 + \alpha\int_{\eta_1}^{\eta_2}\|u(\eta)\|_W^2\,d\eta + c_3\int_{\eta_1}^{\eta_2}\|A_2 u(\eta)\|_Y^p\,d\eta \leq \tag{2.5}$$

$$\leq \int_{\eta_1}^{\eta_2}\|f(\eta)\|_{Y'}\|A_2 u(\eta)\|_Y\,d\eta + K_1,$$

where K_1 depends only on c_4.

(*) Pervenuta all'Accademia il 23 settembre 1967.

Let us observe, first of all, that, by *a*) and the second of (2.4), it is

$$A_2 u \in L^2(0, T; Z') \cap L^p(0, T; Y). \tag{2.6}$$

On the other hand, by (1.24), the first of (2.4) and hypothesis VI), it results

$$u' \in L^2(0, T; Z) + L^{p'}(0, T; Y'). \tag{2.7}$$

$A_2 u$ and u' belong therefore to spaces which are dual one of the other.

As $u \in L^\infty(0, T; V_2)$ and A_2 is, by definition, a linear, continuous, self adjoint, positive operator from V_2 to V_2', it follows, from a theorem of Strauss [4], that $u(\eta)$ is V_2–continuous in $[0, T]$ and that

$$2\int_{\eta_1}^{\eta_2} \langle u'(\eta), A_2 u(\eta)\rangle \, d\eta = \langle u(\eta_2), A_2 u(\eta_2)\rangle - \langle u(\eta_1), A_2 u(\eta_1)\rangle. \tag{2.8}$$

In order to prove (2.5), we consider the scalar product of (1.24) by $A_2 u(\eta)$ (this scalar product has obviously sense, bearing in mind the hypotheses made and what has just been proved). We obtain, in this way, the relation

$$\begin{aligned} &\langle u'(\eta), A_2 u(\eta)\rangle + \langle (A_1 + A_3) u(\eta), A_2 u(\eta)\rangle + \\ &+ \langle B A_2 u(\eta), A_2 u(\eta)\rangle = \langle f(\eta), A_2 u(\eta)\rangle, \end{aligned} \tag{2.9}$$

which, after integration from η_1 to η_2, becomes, by (2.8),

$$\begin{aligned} &\frac{1}{2}\langle u(\eta_2), A_2 u(\eta_2)\rangle - \frac{1}{2}\langle u(\eta_1), A_2 u(\eta_1)\rangle + \\ &+ \int_{\eta_1}^{\eta_2} \langle (A_1 + A_3) u(\eta), A_2 u(\eta)\rangle \, d\eta + \int_{\eta_1}^{\eta_2} \langle B A_2 u(\eta), A_2 u(\eta)\rangle \, d\eta = \\ &= \int_{\eta_1}^{\eta_2} \langle f(\eta), A_2 u(\eta)\rangle \, d\eta . \end{aligned} \tag{2.10}$$

We now observe that, by condition III),

$$\langle (A_1 + A_3) v, A_2 v\rangle = ((A_1 + A_3) v, A_2 v)_H \geq \alpha \| v \|_W^2, \qquad \forall v \in D(A_1 A_2). \tag{2.11}$$

From a density theorem for interpolation spaces (Lions [3]) it follows that $D(A_1 A_2)$ is dense in W; it is therefore, by (2.11), (2.4) and condition IV),

$$\langle (A_1 + A_3) v, A_2 v\rangle \geq \alpha \| v \|_W^2 \qquad \forall v \in W. \tag{2.12}$$

Introducing finally (2.12) into (2.10), we obtain, bearing in mind (1.22),

$$(2.13)\quad \frac{1}{2}\langle u(\eta_2), A_2 u(\eta_2)\rangle - \frac{1}{2}\langle u(\eta_1), A_2 u(\eta_1)\rangle + \alpha \int_{\eta_1}^{\eta_2} \|u(\eta)\|_W^2 \, d\eta +$$

$$+ c_3 \int_{\eta_1}^{\eta_2} \|A_2 u(\eta)\|_Y^p \, d\eta \leq \int_{\eta_1}^{\eta_2} \|f(\eta)\|_{Y'} \|A_2 u(\eta)\|_Y \, d\eta + K_1,$$

and this relation, by (1.11), coincides with (2.5).

LEMMA 3: *Assume that hypotheses* I), II), III), IV), VI), VIII), X) *are verified and let* $u(\eta)$ *be a solution in* $[0, T]$ *of* (1.24), *with* $f \in L^{p'}(0, T; Y')$. *There exists then a number* $\varepsilon > 0$ *such that*

$$(2.14)\qquad A_2 u \in H^{\varepsilon}(0, T; D(A_1^{\varepsilon})).$$

Let us observe, first of all, that

$$(2.15)\qquad Z = [D(A_2), D(A_1)']_{1/2} \subseteq [H, D(A_1)']_{1/2} = D(A_1^{1/2})'.$$

It follows then from (2.7) that, by hypothesis X),

$$(2.16)\quad u' \in L^2(0, T; Z) + L^{p'}(0, T; Y') \subseteq L^2(0, T; D(A_1^{1/2})') +$$

$$+ L^{p'}(0, T; D(A_1^{s})') \subseteq L^{p'}(0, T; D(A_1^{\rho})')$$

where $\rho = \max(1/2, s)$.

Being $u \in L^2(0, T; W) \subset L^{p'}(0, T; D(A_1^{\rho})')$, it results therefore

$$(2.17)\qquad u \in H^{1,p'}(0, T; D(A_1^{\rho})').$$

Bearing in mind conditions I) and II) and the observations made in § 1 on the permutability of operators, we know, on the other hand, that

$$(2.18)\qquad D(A_1^{\sigma+\rho}) \subseteq D(A_2 A_1^{\rho}) = D(A_1^{\rho} A_2).$$

Consequently, A_2 is a linear, continuous operator from $D(A_1^{\sigma+\rho})$ to $D(A_1^{\rho})$. As A_2 is self adjoint and each one of the spaces $D(A_1^{\sigma+\rho})$, $D(A_1^{\rho})$, H is dense in the following, it follows, by transposition, that A_2 is linear and continuous from $D(A_1^{\rho})'$ in $D(A_1^{\sigma+\rho})'$. It is then, by (2.17),

$$(2.19)\qquad A_2 u \in H^{1,p'}(0, T; D(A_1^{\sigma+\rho})').$$

Moreover, by an embedding theorem of Sobolev (see Nikolskii [5]), setting $\nu = \sigma + \rho$, it results

$$(2.20)\quad H^{1,p'}(0, T; D(A_1^{\nu})') \subseteq H^{\frac{3}{2}-\frac{1}{p'}}(0, T; D(A_1^{\nu})') \subset H^{1/2}(0, T; D(A_1^{\nu})').$$

Hence

$$(2.21)\qquad A_2 u \in H^{1/2}(0, T; D(A_1^{\nu})').$$

Observing that, by hypothesis I),

(2.22) $$Z' = [D(A_1), D(A_2)']_{1/2} \subseteq [D(A_1), D(A_1^{\sigma})']_{1/2} = D(A_1^{\frac{1-\sigma}{2}}),$$

it follows, from (2.6), that

(2.23) $$A_2 u \in L^2(0, T; Z') \subseteq L^2(0, T; D(A_1^{\frac{1-\sigma}{2}})).$$

Utilizing an interpolation theorem of Lions [6], we obtain then, from (2.21), (2.23), $\forall \vartheta$ with $0 < \vartheta < 1$,

(2.24) $$A_2 u \in L^2(0, T; D(A_1^{\frac{1-\sigma}{2}})) \cap H^{1/2}(0, T; D(A_1^{\nu})') \subset$$
$$\subset [L^2(0, T; D(A_1^{\frac{1-\sigma}{2}})), H^{1/2}(0, T; D(A_1^{\nu})')]_{\vartheta} =$$
$$= H^{\vartheta/2}(0, T; [D(A_1^{\frac{1-\sigma}{2}}), D(A_1^{\nu})']_{\vartheta}) = H^{\vartheta/2}(0, T; D(A_1^{\mu})),$$

having set $\mu = \frac{1-\sigma}{2} - \vartheta\left(\nu + \frac{1}{2} - \frac{\sigma}{2}\right)$.

Therefore, if $\vartheta = \frac{1-\sigma}{2\nu + 2 - \sigma}$ and $\varepsilon = \frac{\vartheta}{2}$, (2.24) reduces to (2.14) and the lemma is proved.

Let us now give the proof of Theorem 1.

Let $\{h_j\}$ be a basis on $Z' \cap Y$ (which, by IX), is separable); $\{h_j\}$ will then be a basis, by condition IX), also in Z' and in Y. Setting $g_j = G_2 h_j$ (G_2 being, as usual, Green's operator corresponding to A_2) and observing that by the interpolation theorem of Lions which we have already recalled, A_2 is a linear and continuous operator from W on Z' (see lemma 1), the sequence $\{g_j\}$ is a basis in W; as W is dense in V_2 (being $W \supset D(A_1)$ which is dense in V_2), $\{g_j\}$ is a basis also in V_2.

We now set

(2.25) $$u_n(\eta) = \sum_{j=1}^{n} \alpha_{jn}(\eta)\, g_j \quad , \quad A_2 u_n(\eta) = \sum_{j=1}^{n} \alpha_{jn}(\eta)\, h_j$$

and consider the system of ordinary differential equations, associated to (1.24),

(2.26) $$\langle u_n'(\eta), h_j\rangle + \langle (A_1 + A_3) u_n(\eta), h_j\rangle + \langle BA_2 u_n(\eta), h_j\rangle = \langle f(\eta), h_j\rangle$$
$$(j = 1, \cdots, n).$$

Multiplying (2.26) by $\alpha_{jn}(\eta)$ and summing, we obtain, by (2.25),

(2.27) $$\langle u_n'(\eta), A_2 u_n(\eta)\rangle + \langle (A_1 + A_3) u_n(\eta), A_2 u_n(\eta)\rangle +$$
$$+ \langle BA_2 u_n(\eta), A_2 u_n(\eta)\rangle = \langle f(\eta), A_2 u_n(\eta)\rangle.$$

From (2.27) it follows (applying the same procedure used in lemma 2 to obtain (2.5) from (2.9)) that, $\forall \eta \in (0, T]$,

$$\frac{1}{2} \| u_n(\eta) \|_{V_2}^2 + \alpha \int_0^\eta \| u_n(t) \|_W^2 \, dt + c_3 \int_0^\eta \| A_2 u_n(t) \|_Y^p \, dt \leq \tag{2.28}$$

$$\leq \frac{1}{2} \| u_n(0) \|_{V_2}^2 + \int_0^\eta \| f(t) \|_{Y'} \| A_2 u_n(t) \|_Y \, dt + K_1 \leq$$

$$\leq \frac{1}{2} \| u_n(0) \|_{V_2}^2 + \left\{ \int_0^\eta \| f(t) \|_{Y'}^{p'} \, dt \right\}^{1/p'} \left\{ \int_0^\eta \| A_2 u_n(t) \|_Y^p \, dt \right\}^{1/p} + K_1 .$$

Let $u_{0,n}$ be a linear combination of $g_1, \cdots, g_n$ such that

$$\lim_{n \to \infty} u_{0,n} \underset{V_2}{=} u_0 . \tag{2.29}$$

Setting $u_n(0) = u_{0,n}$, from (2.28) we obtain the relations

$$\operatorname{Sup}_{\eta \in [0,T]} \| u_n(\eta) \|_{V_2} \leq K_2 \quad , \quad \int_0^T \| u_n(\eta) \|_W^2 \, d\eta \leq K_3 , \tag{2.30}$$

$$\int_0^T \| A_2 u_n(\eta) \|_Y^p \, d\eta \leq K_4 ,$$

where the quantities K_j do not depend on n.

The *a priori* estimates (2.30) enable us to state the existence, $\forall n$, in $[0, T]$ of a unique solution of system (2.26) satisfying the initial condition

$$u_n(0) = u_{0,n} . \tag{2.31}$$

By hypothesis VI) and lemma 3, from relations (2.30) it follows that

$$\int_0^T \| B A_2 u_n(\eta) \|_{Y'}^{p'} \, d\eta \leq K_5 \quad , \quad \| A_2 u_n \|_{H^\varepsilon(0,T;D(A_1^\varepsilon))} \leq K_6 \qquad (\varepsilon > 0). \tag{2.32}$$

It is therefore possible to extract from $\{u_n(\eta)\}$ a subsequence (which will also be denoted $\{u_n(\eta)\}$) such that

$$\lim_{n \to \infty}{}^{**} u_n \underset{L^\infty(0,T;V_2)}{=} u \quad , \quad \lim_{n \to \infty}{}^{*} u_n \underset{L^2(0,T;W)}{=} u \quad , \quad \lim_{n \to \infty}{}^{*} A_2 u_n \underset{L^p(0,T;Y)}{=} A_2 u , \tag{2.33}$$

the notation lim** standing for the limit in the weak* topology. Moreover, by the second of (2.32), as the embedding of $H^\varepsilon(0, T; D(A_1^\varepsilon))$ in $L^2(0, T; H)$ is completely continuous (having assumed that the embedding of $V_1 = D(A_1^{1/2})$ in H is completely continuous),

$$\lim_{n \to \infty} A_2 u_n \underset{L^2(0,T;H)}{=} A_2 u . \tag{2.34}$$

Consequently, by hypothesis VII), extracting eventually a further subsequence,

$$\lim_{n\to\infty}{}^{*}\ BA_2\, u_n \underset{L^{p'}(0,T;Y')}{=} BA_2\, u\,. \tag{2.35}$$

From (2.33) it follows immediately that the limit function $u(\eta)$ satisfies condition *a*); by (2.29), (2.31) $u(\eta)$ satisfies also the initial condition (1.25).

Bearing in mind (2.26), (2.33), (2.35), $u(\eta)$ is a solution of the equation

$$\int_0^T \{-\langle u(\eta), \varphi'(\eta)\rangle + \langle (A_1+A_3)\, u(\eta), \varphi(\eta)\rangle + \langle BA_2\, u(\eta), \varphi(\eta)\rangle\}\, d\eta = \int_0^T \langle f(\eta), \varphi(\eta)\rangle\, d\eta\,, \tag{2.36}$$

$\forall \varphi(\eta) = \sum_{\text{finite}} \psi_j(\eta)\, h_j\,,\ \psi_j(\eta) \in \mathfrak{D}(0, T)$. As the space of these functions is dense in $\mathfrak{D}(0, T; Z'\cap Y)$, it follows therefore that also condition *b*) is verified. $u(\eta)$ is then a solution in $[0, T]$ and the existence theorem is proved.

Let us now prove that such a solution is unique.

Let $u_1(\eta)$ and $u_2(\eta)$ be two solutions in $[0, T]$ such that $u_1(0) = u_2(0)$. If we set $w(\eta) = u_1(\eta) - u_2(\eta)$, $w(\eta)$ is obviously a solution of the equation

$$w'(\eta) + (A_1 + A_3)\, w(\eta) + BA_2\, u_1(\eta) - BA_2\, u_2(\eta) = 0 \tag{2.37}$$

and the following relation, analogous to (2.10), holds

$$\frac{1}{2}\langle w(\eta), A_2 w(\eta)\rangle - \frac{1}{2}\langle w(0), A_2 w(0)\rangle + \int_0^\eta \langle (A_1+A_3)\, w(t), A_2 w(t)\rangle\, dt + \int_0^\eta \langle BA_2\, u_1(t) - BA_2\, u_2(t), A_2\, w(t)\rangle\, dt = 0\,. \tag{2.38}$$

Bearing in mind (1.23) and (2.12) and that $w(0) = 0$, we obtain, from (2.38),

$$\frac{1}{2}\|w(\eta)\|^2_{V_2} + \alpha\int_0^\eta \|w(t)\|^2_W\, dt \le c_5 \int_0^\eta \|A_2\, w(t)\|^2_H\, dt = c_5 \int_0^\eta \|w(t)\|^2_{D(A_2)}\, dt\,. \tag{2.39}$$

Observe now that the embedding of W in $D(A_2)$ is completely continuous. In fact, by hypotheses I) and II),

$$D((A_1 A_2)^{\frac{1}{2}}) = D(A_1^{\frac{1-\sigma}{2}} A_1^{\frac{\sigma}{2}} A_2^{\frac{1}{2}}) \subseteq D(A_1^{\frac{1-\sigma}{2}} A_2)\,. \tag{2.40}$$

On the other hand,

$$D(A_1^{\frac{1-\sigma}{2}} A_2) = G_2\, D(A_1^{\frac{1-\sigma}{2}}) \quad , \quad D(A_2) = G_2\, H \tag{2.41}$$

and, by a theorem of complete continuity for interpolation spaces (Lions and Peetre [7]) the embedding of $D(A_1^{\frac{1-\sigma}{2}})$ in H is completely continuous, being $\sigma < 1$. Hence also the embedding of $G_2 D(A_1^{\frac{1-\sigma}{2}})$ in $G_2 H$ is completely continuous and consequently by (2.40) also that of W in $D(A_2)$.

The following relation (Lions [8]) then holds, $\forall \varepsilon > 0$,

$$\| v \|_{D(A_2)} \leq \varepsilon \| v \|_W + c_3 \| v \|_{V_2} ; \tag{2.42}$$

choosing $\varepsilon = \frac{\alpha}{2 c_5}$, we obtain, by (2.39),

$$\frac{1}{2} \| w(\eta) \|_{V_2}^2 + \frac{\alpha}{2} \int_0^\eta \| w(t) \|_W^2 \, dt \leq c_6 \int_0^\eta \| w(t) \|_{V_2}^2 \, dt . \tag{2.43}$$

From (2.43) follows that $w(\eta) = 0$ in $[0, T]$, which proves the uniqueness theorem.

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