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Almost-periodic solutions of the equation of Schrödinger type. Nota II

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente BENIAMINO SEGRE

NOTE DI SOCI

Analisi matematica. — *Almost-periodic solutions of the equation of Schrödinger type.* Nota [(*)] II del Corrisp. LUIGI AMERIO.

RIASSUNTO. — Si assegnano delle condizioni perché l'equazione del tipo di Schrödinger, con operatore e termine noto quasi-periodici, abbia una soluzione, quasi-periodica e delle condizioni perché le autosoluzioni dell'equazione omogenea, con operatore periodico siano quasi-periodiche.

2. PROOF OF THEOREM I.—*Existence.* Let $\{x_n(t)\}$ be a sequence minimizing the functional $\mu(x)$ in Γ_z: precisely

$$\mu(x_n) \downarrow \tilde{\mu}. \tag{2,1}$$

Since the sequence $\{x_n(0)\}$ is bounded, we assume, forthwith, that it converges weakly: hence

$$\lim_{n\to\infty}{}^{*}\, x_n(0) = \tilde{x}(0)$$

and, $\forall p$,

$$\lim_{n\to\infty}{}^{*}\, x_{p+n}(0) = \tilde{x}(0). \tag{2,2}$$

Let us now observe that, if $\rho_k \geq 0$, $\sum_1^q{}_k \rho_k = 1$, the function

$$w(t) = \sum_1^q{}_k \rho_k x_{p+k}(t)$$

(*) Pervenuta all'Accademia il 21 ottobre 1967.

satisfies (1,1) and it results, by (2,1),

$$\mu(w) \leq \sum_1^q {}_k\, \rho_k\, \mu(x_{p+k}) \leq \mu(x_{p+1}) .$$

Hence we can construct, $\forall p$, by a theorem of Mazur, a solution

$$w_p(t) = \sum_1^{q_p} {}_k\, \rho_{pk}\, x_{p+k}(t) \qquad \left(\rho_{pk} \geq 0 \,,\ \sum_1^{q_p} {}_k\, \rho_{pk} = 1\right)$$

such that

$$\| w_p(0) - \tilde{x}(0) \| \leq \frac{1}{p} \quad , \quad \mu(w_p) \downarrow \tilde{\mu} .$$

Therefore we assume, forthwith:

$$\lim_{n \to \infty} x_n(0) = \tilde{x}(0) . \tag{2,3}$$

Let $\tilde{x}(t)$ be the solution corresponding to the initial value $\tilde{x}(0)$. By formula (1,9), of continuous dependence on data, it follows, uniformly on every bounded interval,

$$\lim_{n \to \infty} x_n(t) = \tilde{x}(t) . \tag{2,4}$$

Hence, by (1,12), (1,13) and (2,1),

$$\varphi(\tilde{x}\,; v\,, \tau) \leq \varphi(z\,; v\,, \tau) \quad , \quad \mu(\tilde{x}) \leq \tilde{\mu} .$$

Therefore $\tilde{x}(t) \in \Gamma_z$ and it is, necessarily, $\mu(\tilde{x}) = \tilde{\mu}$.

Uniqueness. Let $\tilde{x}(t)$ and $x_0(t)$ be two different solutions $\in \Gamma_z$ such that

$$\mu(\tilde{x}) = \mu(x_0) = \tilde{\mu} . \tag{2,5}$$

Then $u(t) = \tilde{x}(t) - x_0(t)$ satisfies the homogeneous equation and we have, because of the continuous embedding of X in Y and the fundamental formula a'):

$$\operatorname*{Inf}_J \| \tilde{x}(t) - x_0(t) \| \geq \frac{1}{\sigma} \operatorname*{Inf}_J \| \tilde{x}(t) - x_0(t) \|_Y = \frac{1}{\sigma} \| \tilde{x}(0) - x_0(0) \|_Y = \vartheta > 0 \tag{2,6}$$

(it cannot be $\vartheta = 0$, by the uniqueness of the solution for the initial value problem).

There follows, $\forall t \in J$,

$$\| \tilde{x}(t) - x_0(t) \| \geq \frac{\vartheta}{\tilde{\mu}} \max \{ \| \tilde{x}(t) \| \,, \| x_0(t) \| \}$$

and, by the theorem of parallelogram,

$$\left\| \frac{\tilde{x}(t) + x_0(t)}{2} \right\| \leq \sqrt{1 - \frac{\vartheta^2}{4\tilde{\mu}^2}} \max \{ \| \tilde{x}(t) \| , \| x_0(t) \| \} \leq \sqrt{1 - \frac{\vartheta^2}{4\tilde{\mu}^2}}\ \tilde{\mu}$$

that is

$$\mu\left(\frac{\tilde{x}+x_0}{2}\right) < \tilde{\mu} ,$$

which is absurd.

The proof of the corollary follows immediately from the uniqueness of the minimal solution. In fact, *if* A (t) *and* $f(t)$ *are periodic, with period* ω, *then* $\tilde{x}(t)$ *and* $\tilde{x}(t+\omega)$ *are minimal solutions, in* Γ_z: *hence* $\tilde{x}(t) = \tilde{x}(t+\omega)$.

3. PROOF OF THEOREM II.—α) Since, by hypothesis, $z(t)$ is w.u.c., the minimal solution $\tilde{x}(t)$ is w.u.c. too (cfr. observation II, § 1). One proves then that $\tilde{x}(t)$ is w.a.p. by the same method followed in [1], which extends Favard's procedure for linear ordinary a.p. systems.

Moreover, it is obvious that $\tilde{x}(t)$ is Y–w.a.p. We have, in fact, denoting by G Green's operator from Y to X,

$$(\tilde{x}(t), y)_Y = (\tilde{x}(t), Gy) \qquad (\forall y \in Y) \tag{3,1}$$

and the second term is a.p.

β) Let us now prove that $\tilde{x}(t)$ is Y–a.p. For this we shall use the fundamental formula a). Since $f(t)$ is a.p. and $\tilde{x}(t)$ is w.a.p., the scalar product $(f(t), \tilde{x}(t))$ is a.p. It follows, by a), that the derivative $\frac{d}{dt}\|\tilde{x}(t)\|_Y^2$ is a.p. Hence $\|\tilde{x}(t)\|_Y^2$ is a.p., since it is bounded.

Let us now consider an arbitrary sequence $\{s_n\}$ and the sequences $\{A(t+s_n)\}$, $\{f(t+s_n)\}$, $\{\tilde{x}(t+s_n)\}$. We can assume, because of almost-periodicity, that it results uniformly,

$$\begin{aligned} &\lim_{n\to\infty} A(t+s_n) = A_s(t) \quad , \quad \lim_{n\to\infty} A'(t+s_n) = A'_s(t) , \\ &\lim_{n\to\infty} f(t+s_n) = f_s(t) \quad , \quad \lim_{n\to\infty} f'(t+s_n) = f'_s(t) . \end{aligned} \tag{3,2}$$

Moreover, also uniformly,

$$\lim_{n\to\infty}{}^{*} \tilde{x}(t+s_n) = \tilde{x}_s(t) ,$$

where $\tilde{x}_s(t)$ is, by (1,1), a w.a.p. solution of the equation

$$\int_J \{i(x(t), h'(t))_Y + (A_s(t)\, x(t) + f_s(t), h(t))\}\, dt = 0 . \tag{3,3}$$

Hence the norm $\|\tilde{x}_s(t)\|_Y$ is a.p.

By a criterion of almost-periodicity [2] (certainly valid in Hilbert spaces), we then deduce that $\tilde{x}(t)$ is Y–a.p.

γ) Let us now prove that, if the bounded solution $z(t)$ is u.c., $\tilde{x}(t)$ is a.p.

It follows in fact from (1,1), $\forall v \in X$,

$$i\frac{d}{dt}(\tilde{x}(t), v)_Y = (A(t)\, \tilde{x}(t) + f(t), v) ,$$

that is

$$(3,4) \qquad i\,(\overset{*}{x}(t)\,,\,v)_Y = i\,(\overset{*}{x}(0)\,,\,v)_Y + \left(\int_0^t \{A(\eta)\,\overset{*}{x}(\eta) + f(\eta)\}\,d\eta\,,\,v\right).$$

We obtain moreover, by (3,1),

$$(\overset{*}{x}(t)\,,\,v)_Y = (G\overset{*}{x}(t)\,,\,v)$$

and therefore, by (3,4),

$$(3,5) \qquad iG\overset{*}{x}(t) = iG\overset{*}{x}(0) + \int_0^t \{A(\eta)\,\overset{*}{x}(\eta) + f(\eta)\}\,d\eta\,.$$

Since it results

$$\| G\overset{*}{x}(t+\tau) - G\overset{*}{x}(t)\| \leq \|G\|\,\|\overset{*}{x}(t+\tau) - \overset{*}{x}(t)\|_Y\,,$$

$G\overset{*}{x}(t)$ is an a.p. function.

Hence the integral

$$\int_0^t \{A(\eta)\,\overset{*}{x}(\eta) + f(\eta)\}\,d\eta$$

is a.p.

By observation II, § 1, the minimal solution $\overset{*}{x}(t)$ is u.c. Hence $A(t)\,\overset{*}{x}(t) + f(t)$ is u.c. and, therefore, a.p.

Since $f(t)$ is a.p., it follows that $A(t)\,\overset{*}{x}(t)$ is a.p.

Let us now observe that it is, by (1,2) and since $A(t)$ is $\mathfrak{A}$–a.p.:

$$0 < \nu \leq \|A(t)\|_{\mathfrak{A}} \leq N < +\infty\,.$$

Hence there exists the inverse operator $A^{-1}(t)$ and we have [3]

$$N^{-1} \leq \|A^{-1}(t)\|_{\mathfrak{A}} \leq \nu^{-1}\,.$$

It is, moreover,

$$\|A^{-1}(t+\tau) - A^{-1}(t)\|_{\mathfrak{A}} = \|A^{-1}(t+\tau)\,(A(t) - A(t+\tau))\,A^{-1}(t)\|_{\mathfrak{A}} \leq$$
$$\leq \nu^{-2}\,\|A(t+\tau) - A(t)\|_{\mathfrak{A}}\,.$$

Therefore $A^{-1}(t)$ is $\mathfrak{A}$–a.p. and the function $\overset{*}{x}(t) = A^{-1}(t)\,A(t)\,\overset{*}{x}(t)$ is a.p. too.

δ) *Assume*, at last, $A(t) = I$. In this case we can prove that, *if $f(t)$ and $f'(t)$ are a.p. and if there exists a bounded solution*, $z(t)$, *then the minimal solution*, $\overset{*}{x}(t)$, *is a.p.*.

We shall prove the thesis, by using Zaidman's procedure [4], for the same question concerning paper [1].

In order to prove that $\overset{*}{x}(t)$ is w.a.p., observe, at first, that every solution $x(t)$ can be put, by the fundamental formula b'), in the form

$$(3,6) \qquad x(t) = U(t)\,x(0) + w(t)\,,$$

where $U(t)\,x(0)$ satisfies the homogeneous equation, with initial value $x(0)$, and $w(t)$ satisfies (1,1), with initial value $w(0) = 0$; $U(t)$ is a X-strongly-continuous and unitary operator.

Let now $\{s_n\}$ be an arbitrary sequence: we may assume, forthwith, that it results

$$\lim_{n\to\infty}{}^{*}\ \tilde{x}(s_n) = \tilde{x}_s(0) \tag{3,7}$$

and that (3,2) are valid.

Since $\tilde{x}(t+s_n)$ satisfies the equation with operator I and known term $f(t+s_n)$, we obtain, by (3,6),

$$\tilde{x}(t+s_n) = U(t)\,\tilde{x}(s_n) + w_n(t) \qquad (w_n(0) = 0).$$

Moreover, $\forall v \in X$,

$$(U(t)\,\tilde{x}(s_n)\,,\,v) = (\tilde{x}(s_n)\,,\,U^{-1}(t)\,v)\,.$$

Hence, by (3,7) and because of the continuity of $U^{-1}(t)\,v$, the sequence $\{U(t)\,\tilde{x}(s_n)\}$ is weakly convergent, uniformly on every bounded interval $-T \le t \le T$.

Moreover, by (1,9), on the same interval,

$$\|w_n(t) - w_m(t)\| \le M_T \left\{ \|f(s_n) - f(s_m)\| + \int_{-T}^{T} \|f'(\eta + s_n) - f'(\eta + s_m)\|\,d\eta \right\},$$

that is the sequence $\{w_n(t)\}$ converges uniformly on $-T \le t \le T$.

It follows the uniform weak convergence of $\{\tilde{x}(t+s_n)\}$ on every bounded interval.

One proves then, as in [1], that the weak convergence is uniform on J (which implies $\tilde{x}(t)$ w.a.p.) and that the range $R_{\tilde{x}(t)}$ is r.c.: hence $\tilde{x}(t)$ is a.p. [5].

4. PROOF OF THEOREM IV.—Let us assume that $A(t)$ is a *periodic* operator, with period 1. Let $u(t)$ be an eigensolution of (1,3): then also $u(t+n) - u(t+m)$, $\forall$ integers n and m, satisfies the same equation.

Hence, by a'),

$$\|u(t+n) - u(t+m)\|_Y = \|u(n) - u(m)\|_Y\,. \tag{4,1}$$

Let us now assume that $u(t)$ is bounded. Since the sequence $\{u(n)\}$ is bounded and the embedding of X in Y is compact (by hypothesis), $\{u(n)\}$ results Y–r.c.

Setting now, in the space $K = C(J;Y) \cap L^{\infty}(J;Y)$ (cfr. observation III, § 1)

$$\tilde{u}(s) = \{u(t+s)\,;\,t \in J\}\,,$$

we obtain, by (4,1),

$$\|\tilde{u}(n) - \tilde{u}(m)\|_K = \sup_J \|u(t+n) - u(t+m)\|_Y = \|u(n) - u(m)\|_Y\,.$$

Hence the sequence $\{\tilde{u}(n)\}$ is r.c. and, by δ') at observation III, § 1), we deduce that $u(t)$ is Y–a.p.

Let us now prove that $u(t)$ is w.a.p., that is, $\forall v \in X$, the scalar product $(u(t), v)$ is a.p.

Since it is, $\forall y \in Y$, by (3,1),

$$(u(t), y)_Y = (u(t), Gy),$$

the function $(u(t), Gy)$ is a.p. Moreover the set GY is dense on X. Assume, in fact, the contrary is true: then there exists $x_0 \in X$ and $\neq 0$ such that $(x_0, Gy) = 0$, $\forall y \in Y$. Hence $(x_0, Gx_0) = 0$, that is $\|x_0\|_Y = 0$, $x_0 = 0$, which is absurd.

Let us now take, arbitrarily, $v \in X$: to every $\varepsilon > 0$ there corresponds $z_\varepsilon \in Y$ such that $\|v - Gz_\varepsilon\| < \varepsilon$.

Setting $M = \operatorname{Sup}_J \|u(t)\|$, it results then

$$|(u(t+\tau) - u(t), v)| \leq |(u(t+\tau) - u(t), Gz_\varepsilon)| + 2\varepsilon M,$$

that is $(u(t), v)$ is a.p.

Let us assume, at last, that $u(t)$, bounded, is u.c.

In this case we prove that $u(t)$ is a.p. by the same device as at § 3, γ), for proving the almost-periodicity of $\tilde{\tilde{x}}(t)$.

Observation I.—Assume that $A(t) = I$ *and that the embedding of* X *in* Y *is compact: then every eigensolution* $u(t)$ *is a.p.*

It is in fact, by b'), $\|u(t)\| = \|u(0)\|$, that is $\|u(t)\|$ is a.p. Since $u(t)$ is w.a.p., we deduce that it is a.p., by the same criterion used at § 3, β).

Observation II.—Assume that, in the example given at § 1, the open set Ω is *bounded*: then hypothesis 2) of theorem III is satisfied, even if, in (1,5), it results $a(t,\zeta) \geq 0$.

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