Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Periodic or almost-periodic solutions of a non linear functional equation. Nota I

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **43** (1967), n.3-4, p. 161–167.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1967_8_43_3-4_161_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1967.

Analisi matematica. — *Periodic or almost-periodic solutions of a non linear functional equation* (*). Nota I (**) di GIOVANNI PROUSE, presentata dal Corrisp. L. AMERIO.

SUNTO. — Si studiano alcuni problemi di propagazione per un'equazione funzionale non lineare. Dopo un'analisi preliminare del problema di Cauchy, si danno, in particolare, teoremi di esistenza ed unicità delle soluzioni periodiche e quasi-periodiche.

1. In the present paper we shall study some evolution problems for the non linear functional equation

$$u'(\eta) + (A_1 + A_3)\, u(\eta) + BA_2\, u(\eta) = f(\eta), \tag{1.1}$$

where A_1 and A_2 are unbounded positive linear operators, B is a " dissipative " non linear operator and A_3 is a linear operator which is " smaller " (in an appropriate sense) than A_1. Moreover, A_1 and A_2 are permutable, while $\eta \in [0, T]$ or $(-\infty, +\infty)$.

Under the hypotheses we shall give later on in this paragraph, we shall prove an existence and uniqueness theorem of the solution (in the large) of the Cauchy problem, essential introduction to the study of periodic and almost-periodic (a.p.) solutions, which is the main aim of this paper.

An example of a type of equation for which the results we shall give hold is the functional equation

$$\frac{\partial u(x,\eta)}{\partial \eta} - Au(x,\eta) + BA^{\sigma} u(x,\eta) = f(x,\eta) \qquad (x = \{x_1, \cdots, x_m\}),$$

where A is a self adjoint second order elliptic operator and $0 \leq \sigma < 1$; this example will be considered in detail and generalized in § 5.

We wish to recall that Lions and Strauss [1] have proved an existence and uniqueness theorem of the solution of the Cauchy problem for the equation

$$u'(\eta) + A(\eta)\, u(\eta) + B(\eta)\, u(\eta) = f(\eta), \tag{1.2}$$

which, formally, contains (1.1). If $A_2 = I$, then equation (1.1) is obviously a particular case of (1.2) and the existence and uniqueness theorem we shall prove is included in that of Lions and Strauss.

In the general case ($A_2 \neq I$), the proof of the existence and uniqueness of the solution of the Cauchy problem is analogous to that given in [1], but differs from it in the part that refers to the non linear operator B. One of the

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(**) Pervenuta all'Accademia il 23 settembre 1967.

fundamental hypotheses we shall make on B (analogous to the corresponding hypothesis made by Lions and Strauss) is, in fact, that of being "asymptotically monotone"; while it may be observed that, in equation (1.2), it is always possible, by means of the transformation $u(\eta) = e^{k\eta} v(\eta)$, to assume that B is "strictly monotone", this may not be done in the case of equation (1.1) (at least in the case we shall consider). The procedure followed by Lions and Strauss, based on the "strict monotonicity" of B will be substituted by a compactness method, direct consequence of Lemma 3, which we shall prove in § 2.

We observe that it is possible to write (1.1) in an equivalent form. Multiplying (1.1) by A_2, setting $A_2 u(\eta) = v(\eta)$, $A_2 f(\eta) = g(\eta)$ and bearing in mind that the operators A_1 and A_2 are permutable, we obtain

$$v'(\eta) + A_1 v(\eta) + A_2 A_3 G_2 v(\eta) + A_2 Bv(\eta) = g(\eta), \tag{1.3}$$

where G_2 is Green's operator corresponding to A_2.

Equation (1.3) is formally contained in (1.2) and the observations concerning (1.1) hold also for this equation. In what follows, we shall however always consider equation (1.1); the results we shall obtain could anyway be proved substituting (1.1) with (1.3).

Let us now give some fundamental definitions and hypotheses.

If $\mathcal{H}_1$ and $\mathcal{H}_2$ are any two Hilbert spaces, with $\mathcal{H}_1 \subset \mathcal{H}_2$ (1) and dense in $\mathcal{H}_2$, we shall indicate with $[\mathcal{H}_1, \mathcal{H}_2]_\theta$ $(0 < \theta < 1)$ the Hilbert space $\mathcal{H}_1^{1-\theta} \mathcal{H}_2^{\theta}$, intermediate between $\mathcal{H}_1$ and $\mathcal{H}_2$, defined according to the procedure given by Lions in [2]. If K is a Banach space, we shall indicate with $L^p(0, T; K)$ $(p \geq 1)$ the space of functions $v(\eta)$ which are L^p over $(0, T)$ with values in K, i.e. such that

$$\| v \|_{L^p(0,T;K)} = \left\{ \int_0^T \| v(\eta) \|_K^p \, d\eta \right\}^{1/p} < +\infty. \tag{1.4}$$

Let V_1, V_2, H be three real Hilbert spaces, with $V_1 \subset V_2 \subseteq H$, the embedding of V_1 in V_2 being completely continuous and each space dense in the following one (2). Moreover, let $a_1(u, v)$, $a_2(u, v)$ be two bilinear hermitian forms such that there exist two positive constants, c_1 and c_2 for which

$$|a_i(u, v)| \leq c_1 \|u\|_{V_i} \|v\|_{V_i} \qquad \forall u, v \in V_i, \tag{1.5}$$

$$a_i(v, v) \geq c_2 \|v\|_{V_i}^2 \qquad \forall v \in V_i \qquad (i = 1, 2). \tag{1.6}$$

(1) By the notation $\mathcal{H}_1 \subset \mathcal{H}_2$ we mean that $\mathcal{H}_1$ is contained in $\mathcal{H}_2$ and that the inclusion mapping of $\mathcal{H}_1$ into $\mathcal{H}_2$ is continuous.

(2) The complete continuity of V_1 in V_2 is not essential for the existence and uniqueness of the solution of the Cauchy problem when $V_2 = H$ (which corresponds to the case $A_2 = I$). This hypothesis is, in fact, not necessary whenever the non linear operator B is "strictly monotone" as, in such a case, the procedure outlined by Lions and Strauss can be followed.

It is then possible to define in H two linear, closed, unbounded operators A_1 and A_2, with domain $D(A_1)$ and $D(A_2)$ respectively, in the following way. An element $u \in V_i$ belongs to $D(A_i)$ if the linear functional $v \to a_i(u, v)$ is continuous on V_i with the topology induced by H. *The operators* A_i *are therefore defined by the relations*

$$a_i(u, v) = (A_i u, v)_H \qquad \forall v \in V_i \qquad (i = 1, 2) \tag{1.7}$$

and, for the hypotheses made, *are linear, self-adjoint and positive.* Defining in $D(A_i)$ the scalar product by the relation

$$(u, v)_{D(A_i)} = (A_i u, A_i v)_H \text{ (3)}, \tag{1.8}$$

$D(A_i)$ *is a Hilbert space, dense in* H.

It is then possible to introduce the operators A_i^ν, ν real > 0, and it results

$$[D(A_i^r), D(A_i^s)]_\theta = D(A_i^{r(1-\theta)+s\theta}). \tag{1.9}$$

Moreover (Lions [3])

$$V_i = D(A_i^{1/2}) = [D(A_i), H]_{1/2}. \tag{1.10}$$

We shall therefore set, in accordance with (1.8) and (1.9),

$$(u, v)_{D(A_i^\nu)} = (A_i^\nu u, A_i^\nu v)_H. \tag{1.11}$$

By a density theorem of Lions [4], from (1.9) follows that the spaces $D(A_i^\nu)$ are dense in H.

Let us observe that the equations $Au = f$, $Av = g$ have, if the hypotheses made on the forms $a_i(u, v)$ are verified, one and only one solution $\forall f, g \in H$ given respectively by

$$u = G_1 f \in D(A_1) \quad , \quad v = G_2 g \in D(A_2), \tag{1.12}$$

where G_1 and G_2 are Green's operators corresponding to A_1 and A_2 respectively. Furthermore, $D(A_1) = G_1 H$, $D(A_2) = G_2 H$.

Let h be an arbitrary element $\in H$ and consider the equation

$$A_2 A_1 u = h. \tag{1.13}$$

Setting $\tilde{u} = A_1 u$, it results, by the second of (1.12),

$$\tilde{u} = G_2 h$$

and, consequently, by the first of (1.12),

$$u = G_1 G_2 h. \tag{1.14}$$

(3) The norm corresponding to (1.8) is, under the hypotheses made, obviously equivalent to

$$\|v\|_H^2 + \|A_i v\|_H^2.$$

It is, therefore

$$D(A_2 A_1) = G_1 G_2 H . \tag{1.15}$$

Correspondingly, if we consider the equation $A_1 A_2 v = h$, we find that

$$v = G_2 G_1 h \quad , \quad D(A_1 A_2) = G_2 G_1 H . \tag{1.16}$$

From (1.14), (1.15), (1.16) it follows that, if the operators G_1 and G_2 are permutable, also the operators A_1 and A_2 are permutable, being

$$D(A_1 A_2) = D(A_2 A_1) \quad , \quad A_1 A_2 u = A_2 A_1 u \qquad \forall u \in D(A_1 A_2) . \tag{1.17}$$

If, on the other hand, relations (1.17) hold, then, by (1.14), (1.16),

$$G_1 G_2 h = u = G_2 G_1 h \qquad \forall h \in H , \tag{1.18}$$

i.e. the operators G_1 and G_2 are permutable.

All that has been said above applies also to the operators $A_i^\vee$.

Let us now assume that the operators A_1 and A_2 satisfy the following assumptions:

I) *There exists a number* σ, *with* $0 \leq \sigma < 1$, *such that* $D(A_1^\sigma) \subseteq D(A_2)$, *being moreover* $D(A_1^\sigma)$ *dense in* $D(A_2)$;

II) *The operators* A_1 *and* A_2 *are permutable*, i.e. *relations* (1.17) *hold*.

Conditions I) and II) are, for instance, satisfied, if $A_2 = A_1^\sigma$.

Let A_3 *be a continuous, linear operator from* $D(A_1)$ *to* H; we shall assume that:

III) *There exists a positive number* α *such that, setting* $W = D((A_1 A_2)^{1/2})$, *it results*

$$((A_1 + A_3) v , A_2 v)_H \geq \alpha \| v \|_W^2 \qquad \forall v \in D(A_1 A_2) ; \tag{1.19}$$

IV) $A_3 W \subseteq A_1 W$.

In what follows, if K is a Banach space $\subset H$ and dense in H, we shall denote by K′ its dual space, while the space H′ will be identified with H.

Let B *be a non linear operator from a reflexive Banach space* $Y \subset H$, *dense in* H, *to* Y′, *with* $B0 = 0$; we assume that the following hypotheses hold:

V) B *is weakly continuous from finite-dimensional subspaces of* Y to Y′;

VI) *If* $v(\eta) \in L^p(0,T;Y)$ $(p > 2)$, $Bv(\eta) \in L^{p'}(0,T;Y')$ $(1/p + 1/p' = 1)$; *moreover, the map* $v(\eta) \to Bv(\eta)$ *sends bounded sets of* $L^p(0,T;Y)$ *into bounded sets of* $L^{p'}(0,T;Y')$;

VII) *If* $\{v_n\}$ *is an* $L^p(0,T;Y)$*-bounded sequence, such that*

$$\lim_{n \to \infty} v_n \underset{L^2(0,T;H)}{=} v, \tag{1.20}$$

it is possible to extract a subsequence $\{v_{n'}\} \subset \{v_n\}$ *such that*

$$\lim_{n' \to \infty}{}^{*} Bv_{n'} \underset{L^{p'}(0,T;Y')}{=} Bv \text{ (4)}. \tag{1.21}$$

(4) We shall denote by lim* the limit in the weak topology.

VIII) *There exist three positive constants,* c_3 , c_4 , c_5 *such that*

$$\langle Bv , v\rangle \geq c_3 \| v \|_Y^p - c_4 \qquad \forall v \in Y, \tag{1.22}$$

$$\langle Bv_1 - Bv_2 , v_1 - v_2\rangle \geq - c_5 \| v_1 - v_2 \|_H^2 \qquad \forall v_1 , v_2 \in Y. \tag{1.23}$$

Finally, we shall assume that the spaces introduced satisfy the following conditions:

IX) *The space* $[D(A_1) , D(A_2)']_{1/2} \cap Y$ *is separable and dense in* $[D(A_1) , D(A_2)']_{1/2}$ *and in* Y;

X) *There exists a number* $s > 0$ *such that* $Y' \subset D(A_1^s)'$.

We observe that hypotheses III), IV), IX) correspond to those given by Lions and Strauss and also hypotheses V), VI), VIII) are analogous to the conditions imposed by these Authors on the non linear operator B. Hypothesis II) (which is automatically verified if $A_2 = I$) is, on the other hand, essential for the validity of all the lemmas and theorems we shall give. Finally, hypotheses VII) and X) will be used in the proof of lemma 3 which, as we have already mentioned, states a compactness property of the solutions. If, therefore, in relation (1.23), it were $c_5 = 0$, (i.e. if B were " strictly monotone ") it would be possible, in the proof of the existence and uniqueness theorem of the solution of the Cauchy problem, to utilize the procedure given by Lions and Strauss, based on the " strict monotonicity " of B (see footnote (2)); hypotheses VII) and X) would then no longer be necessary.

Let us now consider the equation

$$u'(\eta) + (A_1 + A_3)\, u(\eta) + BA_2\, u(\eta) = f(\eta) \qquad (0 \leq \eta \leq T) \tag{1.24}$$

where f takes its values in Y'. Setting $Z = [D(A_2) , D(A_1)']_{1/2}$, *we shall say that* $u(\eta)$ *is a solution in* $[0 , T]$ *of* (1.24) *if*:

a) $u \in L^\infty(0 , T ; V_2) \cap L^2(0 , T ; W)$, $A_2 u \in L^p(0 , T ; Y)$ $(p > 2)$;

b) $u(\eta)$ *satisfies* (1.24) *almost everywhere in* (0 , T), *in the sense of distributions with values in* $Z + Y'$.

As will be seen, the hypotheses made above are sufficient, if $f \in L^{p'}(0 , T ; Y')$, to guarantee the existence and uniqueness of the solution of the Cauchy problem and the existence of a periodic solution (if $f(\eta)$ is periodic). In order to prove the uniqueness of the periodic solution and the existence and uniqueness of an a.p. solution (assuming that $f(\eta)$ is a.p.) it will be necessary to make the following further hypothesis:

XI) *If* γ *is the embedding constant of* W *in* $D(A_2)$, *it results* $c_5 < \frac{\alpha}{\gamma^2}$, *where* α *and* c_5 *are the constants appearing in relations* (1.19) *and* (1.23).

Let us enounce the theorems which, together with some auxiliary lemmas, will be proved in detail in the following paragraphs.

THEOREM 1: *Assume that all the hypotheses made above (with the exception of* XI)) *are verified and that* $f \in L^{p'}(0, T; Y')$. *There exists then in* $[0, T]$ *one, and only one, solution of* (1.24) *satisfying the initial condition*

$$u(0) = u_0 \tag{1.25}$$

with u_0 *arbitrary element of* V_2.

THEOREM 2: *If the hypotheses of theorem 1 are satisfied and* $f(\eta)$ *is periodic with period* T, *there exists at least one solution which is periodic with period* T.

THEOREM 3: *Assume that all the hypotheses made (except* XI)) *are verified and that* [5]

$$\sup_{t \in J} \left\{ \int_0^1 \|f(t+\eta)\|_{Y'}^{p'} \, d\eta \right\}^{1/p'} = \sup_{t \in J} \|f(t)\|_{L^{p'}(0,1;Y')} = M_0 < +\infty \tag{1.26}$$

$$(J = (-\infty, +\infty)).$$

There exists then in J *at least one solution* $\tilde{u}(\eta)$ *such that*

$$\begin{aligned} &\sup_{\eta \in J} \|\tilde{u}(\eta)\|_{V_2} = M_1 < +\infty \quad , \quad \sup_{t \in J} \|\tilde{u}(t)\|_{L^2(0,1;W)} = M_2 < +\infty , \\ &\sup_{t \in J} \|A_2 \tilde{u}(t)\|_{L^p(0,1;Y)} = M_3 < +\infty \quad , \quad \sup_{t \in J} \|A_2 \tilde{u}(t)\|_{H^\varepsilon(0,1;D(A_1^\varepsilon))} = M_4 < +\infty \end{aligned} \tag{1.27}$$

$$(\varepsilon > 0).$$

Moreover, if also condition XI) *holds, then* $\tilde{u}(t)$ *is the only* $L^2(0, 1; V_2)$-*bounded solution in* J *and, if* $v(\eta)$ *is any other solution corresponding to the same known term, it results*

$$\lim_{\eta \to +\infty} \|\tilde{u}(\eta) - v(\eta)\|_{V_2} = 0 . \tag{1.28}$$

THEOREM 4: *Assume that all hypotheses made hold and that* $f(t)$ *is* $L^{p'}(0, 1; Y')$-*weakly a.p. Then the bounded solution* $\tilde{u}(t)$ *(which, by theorem 3, exists and is unique) is* $L^2(0, 1; V_2)$-*a.p.*

THEOREM 5: *If the assumptions of theorem 4 are verified and* $f(t)$ *is* $L^{p'}(0, 1; Y')$-*a.p., then* $\tilde{u}(\eta)$ *is* V_2-*a.p. and* $\tilde{u}(t)$ *is* $L^2(0, 1; W)$-*a.p.*

As will be shown in § 5, the theorems given above can, in particular, be applied to the following example.

Let Ω be an open, bounded set of the Euclidean space S^m ($x = \{x_1, x_2, \cdots, x_m\}$), with boundary Γ. Using the notations introduced at the beginning of this §, we set $H = L^2(\Omega) = L^2$, $V_1 = H_0^1(\Omega) = H_0^1$ and

$$\begin{aligned} A_1 &= \sum_{j,k=1}^m a_{jk} \frac{\partial^2}{\partial x_j \partial x_k} \quad , \quad A_3 = \sum_{j=1}^m a_j(x) \frac{\partial}{\partial x_j} + a_0(x) , \\ E &= \sum_{j,k=1}^m e_{jk} \frac{\partial^2}{\partial x_j \partial x_k} + e_0 \quad , \quad A_2 = E^\sigma \qquad (0 \le \sigma < 1), \end{aligned} \tag{1.29}$$

(5) In what follows, we shall set $v(t) = \{v(t+\eta); \eta \in [0, 1]\}$ and, consequently,

$$\|v(t)\|_{L^q(0,1;K)} = \left\{ \int_0^1 \|v(t+\eta)\|_K^q \, d\eta \right\}^{1/q} .$$

with $a_{jk} = a_{kj}$, $e_{jk} = e_{kj}$, $a_j(x) \in L^\infty(\Omega)$. We assume moreover that, $\forall v \in H_0^1$

$$(1.30) \qquad \langle A_1 v , v\rangle \geq \mu_1 \| v \|^2_{H_0^1} \quad , \quad \langle Ev , v\rangle \geq \mu_2 \| v \|^2_{H_0^1} \qquad (\mu_1 , \mu_2 > 0) .$$

Let $\beta(\xi)$ be a continuous function, satisfying locally a Lipschitz condition, of the real variable ξ, defined in J and such that $\beta(0) = 0$. We shall suppose that there exists a number $p > 2$ and three positive constants c_6, c_7, c_8 such that

$$(1.31) \qquad c_6 | \xi |^p \leq \xi\beta(\xi) \leq c_7 | \xi |^p \qquad \text{when} \quad |\xi| \geq \bar{\xi},$$

$$(1.32) \qquad \frac{\beta(\xi_1) - \beta(\xi_2)}{\xi_1 - \xi_2} \geq - c_8 \qquad \forall \text{ couple of points } \xi_1, \xi_2 \in J.$$

We consider the equation

$$(1.33) \qquad u'(\eta) + (A_1 + A_3)\, u(\eta) + \beta(E^\sigma u(\eta)) = f(\eta)$$

having set $u(\eta) = \{u(x, \eta) ; x \in \Omega\}$, $u'(\eta) = \left\{\frac{\partial u(x, \eta)}{\partial \eta} ; x \in \Omega\right\}$, $Au(\eta) = \{Au(x, \eta) ; x \in \Omega\}$, $\beta(E^\sigma u(\eta)) = \{\beta(E^\sigma u(x, \eta)) ; x \in \Omega\}$, $f(\eta) = \{f(x, \eta) ; x \in \Omega)\}$.

This equation is obviously a special case of (1.24) and hypotheses I)$\cdots$X) are verified (see § 5) provided the coefficients $a_j(x)$ $(j = 0, 1, \cdots, m)$ are " sufficiently small " and Γ is of class C^{2s}, with $s = \left[\frac{m(p-2)}{4p}\right] + 1$.

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