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# Rendiconti

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# Exceptional singularities of an algebroid surface and their reduction

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# RENDICONTI

DELLE SEDUTE

# DELLA ACCADEMIA NAZIONALE DEI LINCEI

## Classe di Scienze fisiche, matematiche e naturali

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## NOTE DI SOCI

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**Matematica.** — *Exceptional singularities of an algebroid surface and their reduction.* Nota (*) del Socio Straniero OSCAR ZARISKI (**).


RIASSUNTO. — In lavori precedenti l'Autore ha definito il concetto di singolarità eccezionale di una superficie algebrica o algebroide F e ha dato un procedimento canonico per la risoluzione delle singolarità di F nel caso in cui F è priva di singolarità eccezionali. In questa Nota il processo dello scioglimento delle singolarità della F viene completato. L'Autore dà, cioè, un procedimento canonico per la riduzione del massimo $e(F)$ delle molteplicità dei punti singolari eccezionali della F (i quali sono sempre in numero finito).


INTRODUCTION. We deal with a (not necessarily irreducible) algebraic or algebroid surface F, defined over an algebraically closed ground field $k$ of characteristic zero and having the property that locally, at each of its closed points, F can be embedded in the affine 3-space over $k$ (we shall often refer to this property of F by saying that F is an " embedded surface "). If F is algebroid, we are dealing only with the local case, i.e., we assume that F is the spectrum of a complete equidimensional local ring $\mathfrak{o}$, of Krull dimension 2, having $k$ as field of representatives and free from nilpotent elements (other than zero). In the algebroid case, therefore, F has only one closed point (represented by the maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$); this point will be referred to as the center of F.

In [5] we have defined the concept of *equisingularity* on F. If W is an irreducible singular curve of F and Q is a point of W, we know what is

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meant by saying that F is *equisingular, at* Q, *along* W. We recall from [5] that equisingularity of F at Q, along W, implies that (1) F is equimultiple, at Q, along W, and that (2) Q is a simple point of the total singular curve S of F (S = union of the irreducible singular curves of F).

*Definition: A simple point* Q *of* S *such that* F *is equisingular, at* Q, *along the irreducible component of* S *passing through* Q, *is said to be a singular point* of dimensionality type 1. *All other singular points of* F *are called* exceptional *singular points*.

The set of exceptional singularities of F consists therefore of the following points of F: (1) the *isolated* singularities of F (i.e., the singular points which do not lie on singular curves); (2) all the singular points of the total singular curve S of F; (3) certain simple points of S. That the set of exceptional singularities of F is finite follows from Theorem 4.4, part (*b*), of [5], or also from Theorem 5.2, (the Jacobian criterion of equisingularity) of [5]. If F is algebroid then the center O of F is the only possible exceptional singularity of F (since the generic point of any irreducible component W of the singular curve S is not an exceptional singularity).

We have indicated in [3] (Note I, Proposition 5) and have proved in [5] (Theorem 7.4 and Corollary 7.5) that if F has no exceptional singularities, then the normalization $\overline{F}$ of F is non–singular and is obtainable from F by a finite sequence of monoidal transformations $T_i : F_{i+1} \to F_i$ $(i = 0, 1, \cdots, N;$ $F_0 = F ; F_{N+1} = \overline{F})$, such that each $F_i$ is free from exceptional singularities and the center of each $T_i$ is an irreducible singular curve of $F_i$.

For the convenience of the reader we shall state now in some detail the precise facts which underlie the cited theorem 7.4 and which have been brought out in our paper [5] for any embedded surface.

(*a*) Each irreducible component W of the total singular curve of S of F determines an *equivalence class* C (F , W) of singularities of embedded (i.e., plane) algebroid curves (in the sense of our paper [4]), with the property that if Q is *any* point of W which is not an exceptional singularity of F, and if G is *any* non–singular algebroid surface (in the affine 3–space in which F is locally embedded at Q) which is transversal to the curve W, at Q, then the section of F with G is an algebroid *curve* $\Gamma$ (i.e., has no multiple components) and this curve $\Gamma$ has at Q a singularity *which belongs to the equivalence class* C (F , W).

(*b*) Let $C'_1(F, W), C'_2(F, W), \cdots, C'_m(F, W)$ be the set of equivalence classes (of algebroid plane curves) which represent the quadratic transform of C (F , W). Here $m$ is the number of distinct tangent lines of any member $\Gamma$ of the class C (F , W), so that the quadratic transform $\Gamma'$ of $\Gamma$ splits therefore, into $m$ algebroid curves $\Gamma'_1, \Gamma'_2, \cdots, \Gamma'_m$, having distinct centers $O'_1, O'_2, \cdots, O'_m$, and $C'_i(F, W)$ is the equivalence class determined by $\Gamma'_i$. (We note that the $m$ classes $C'_i(F, W)$ need not be distinct). Let $F'$ be the transform of F by the monoidal transformation $T : F' \to F$, with center W, and let $W'$ be the *proper* transform of W on $F'$ (whence $W'$ is a curve, which may be reducible, and each irreducible component of $W'$

corresponds to W). Let Q be, as above, any point of W which is not an exceptional singularity of F. Then the following is true: (1) the total transform of Q on F′ consists exactly of $m$ points $Q'_1, Q'_2, \cdots, Q'_m$, and these points lie on W′; (2) each point $Q'_i$ is a simple point of W′, and if $W'_i$ is the irreducible component of W′ which contains $Q'_i$ then F′ is equisingular at $Q'_i$, along $W'_i$ (the $m$ curves $W'_i$ need not be distinct); (3) for a suitable ordering of the indices we have $C(F', W'_i) = C'_i(F, W)$ $(i = 1, 2, \cdots, m)$; more explicity, if Γ is as in (*a*) and $\Gamma'_1, \Gamma'_2, \cdots, \Gamma'_m$ are as in (*b*), then $\Gamma'_i$ is a section of F′, transversal to $W'_i$ at $Q'_i$.

(*c*) Let F* denote the surface obtained from F by deleting all the exceptional singularities of F. From the fact that $T^{-1}\{Q\}$ is a finite set for any point Q on $F^*$ follows that if we get $F'^* = T^{-1}\{F^*\}$, then $F'^*$ is dominated by the normalization of F* (and is the monoidal transform of F*, with center $W \cap F^*$). Furthermore, also $F'^*$ *is free from exceptional singularities.* In view of the relations $C(F', W'_i) = C'_i(F, W)$ $(i = 1, 2, \cdots, m)$ noted in (*b*), and since any plane algebroid curve can be desingularized by a finite number of successive quadratic transformations, it follows that by a finite number of successive monoidal transformations, centered at singular curves, we obtain the normalization $\bar{F}^*$ of F* and that $\bar{F}^*$ is non–singular.

In particular, if F has no exceptional singularities, then $F^* = F$, $\bar{F}^* = \bar{F}$. Thus, in the absence of exceptional singularities, the problem of reduction of singularities if F is essentially a *problem in dimension* 1: the reduction process consists of and runs parallel to the reduction of the singularities of the curve sections of F which are transversal to the total singular curve S of F. It is for this reason that we say that in this case all the singularities of F* are of dimensionality type 1. The situation is particularly illuminating in the case in which F is a complex-analytic surface. In this case it can be proved (see Whitney [2], § 11–12, and our forthcoming paper [6], § 7) that if Q and W are as in (*a*) and we regard F* as imbedded in affine $\mathbf{A}_3$, locally at Q, in such a way that W is a line in $\mathbf{A}_3$, then the natural vector bundle structure of $\mathbf{A}_3$, over W as base space, induces on F*, in the neighborhood of Q, the structure of fibre bundle over W, the fibre being any curve in the equivalence class C (F,W).

In the general case, F may have exceptional singularities. The reduction of singularities of F can thus be made to depend on the elimination of the exceptional singularities of the surface. The object of this paper is to exhibit an essentially canonical procedure for the elimination of the exceptional singularities of F.

## § 1. REDUCTION TO "QUASI-ORDINARY" MULTIPLE POINTS.

By a *permissible* transformation of F we shall mean a birational regular map $T: F' \to F$ of one of the following types:

(1) A locally quadratic transformation whose center is an *exceptional* singular point of F.

(2) A monoidal transformation whose center is an irreducible singular curve $\Gamma$ of F, provided $\Gamma$ has the following two properties: (2 *a*) if $\Gamma$ is $m$–fold for F, *every* point of $\Gamma$ is $m$–fold for F; (2 *b*) the only singularities of $\Gamma$ are ordinary double points.

It is quite harmless to allow $\Gamma$ to have ordinary double points, because it is easy to see that if O is an ordinary double point of $\Gamma$ and if O is also $m$–fold for F (in accordance with condition (2 *a*)), then the monoidal transformation $T : F' \to F$, with center $\Gamma$, is *locally, at* O, the product $T_1 T_2$ of two permissible monoidal transformations

$$T_1 : F_1 \to F \quad , \quad T_2 : F' \to F_1 ,$$

centered at *regular* algebroid curves. Namely, if $\Gamma_1$ and $\Gamma_2$ are the two branches of $\Gamma$ at O, then $T_1$ is centered at $\Gamma_1$, while $T_2$ is centered at the $T_1^{-1}$–transform $\Gamma_{2,1}$ of $\Gamma_2$ (1).

We note the following consequence: for any point P of $\Gamma$ (including the point $P = O$) the set $T^{-1}\{P\}$ is finite, and hence $F'$ is dominated by the normalization of F (see [5], Proposition 7.2).

We denote by $e(F)$ the maximum of the multiplicities of the exceptional singular points of F, and we set $s = e(F)$. The object of the rest of this paper will be to show that by a finite number of permissible transformations it is possible to transform F into a surface $\overline{F}$ such that $e(\overline{F}) < e(F)$. This will achieve the object stated at the end of the introduction. The actual permissible transformations which we shall have to use in order to reduce the numerical character $e(F)$ will always have $s$–fold center ($s = e(F)$). We note that if $T : F' \to F$ is a permissible transformation and if $F^*$ is the surface obtained from F after deleting the exceptional singularities of F, then the inverse image of $F^*$ on $F'$ is dominated by the normalization of $F^*$ (see Introduction). Therefore, the non–singular transform $F_0$ of F which our reduction process will ultimately lead us to, will be such that the inverse image of $F^*$ on $F_0$ is the normalization of $F^*$.

*Definition* 1.1. *A singular point* P *of* F is called quasi-ordinary *if there exist local transversal parameters* $x_1, x_2$ *of* P *on* F *such that the critical algebroid curve* $\Delta_{x_1 x_2}$ *associated with these parameters has an ordinary double point* (2).

(1) *Proof.* With F embedded in $\mathbf{A}_3$, locally at O, we may assume that $\Gamma$ is defined by the equations $xy = z = 0$. Since $\Gamma$ is $m$–fold, the local equation of F is of the form $\sum_{i=0}^{m} A_i(x, y, z)\, x^i y^i z^{m-i} = 0$, where the $A_i$ are power series in $x, y, z$. Since O is also $m$–fold, we must have $A_0(0, 0, 0) \neq 0$. By the Weierstrass preparation theorem we may then assume that $A_0(x, y, z)$ is 1. If $\mathfrak{o}$ denotes the local ring of O on F, then it follows that $z/xy$ is integral over $\mathfrak{o}$, and hence, locally at O, the $T^{-1}$–transform of F is Spec $\mathfrak{o}[z/xy]$. We set $F_1 = \operatorname{Spec} \mathfrak{o}[z/x]$, $z_1 = \dfrac{z}{x}$, $\mathfrak{o}_1 =$ the local ring of the point $O_1 : x = y = z_1 = 0$ of $F_1$. Then $\Gamma_1$ is the branch $x = z = 0$, $\Gamma_{2,1}$ is the regular arc $y = z_1 = 0$, and $F' = \operatorname{Spec} \mathfrak{o}_1[z_1/y]$. Note that both $\Gamma_{2,1}$ and $O_1$ are $m$–fold for $F_1$.

(2) For the definition of local transversal parameters and of $\Delta_{x_1,x_2}$, see [5], definition 2.3 and the end of section 2.

Note that a quasi-ordinary point is necessarily an exceptional singularity, in view of [5], Theorem 5.2.

The first step of our reduction process will result from the following:

*Proposition 1.2. By successive locally quadratic transformations, centered at exceptional s-fold points, it is possible to obtain a transform* $F_1$ of F *such that either* $e(F_1) < s$, *or* $e(F_1) = s$ *and all exceptional s-fold points of* $F_1$ *are quasi-ordinary.*

*Proof*: Let P be an exceptional singular point of F, of multiplicity $s$. Let $T : F' \to F$ be the locally quadratic transformation of F, with center P. Let P′ be any point of F′ which corresponds to P. We fix a system of local coordinates $x_1, x_2, z$ at P such that $x_1, x_2$ are transversal parameters at P. The local equation of F at P is then of the form

$$(1) \qquad f(x_1, x_2, z) = z^s + A_1(x_1, x_2) z^{s-1} + \cdots + A_s(x_1, x_2) = 0,$$

where $A_i(x_1, x_2)$, is a power series whose initial form is of degree $\geq i$. Let $D(x_1, x_2)$ be the discriminant of $f$ with respect to $z$. From (1) it follows at once that $\frac{x_1}{z}, \frac{x_2}{z}$ cannot be simultaneously zero at P′. Hence either $\frac{x_2}{x_1}, \frac{z}{x_1}$ or $\frac{x_1}{x_2}, \frac{z}{x_2}$ belong to the local ring $\mathfrak{o}'$ of P′ on F′. Let, say, $\frac{x_2}{x_1}, \frac{z}{x_1}$ belong to $\mathfrak{o}'$, and let, say, $a$ and $b$ be the $\mathfrak{m}'$-residues of $\frac{x_2}{x_1}, \frac{z}{x_1}$, where $\mathfrak{m}'$ is the maximal ideal of $\mathfrak{o}'$. Upon replacing $x_2$ and $z$ by $x_2 - ax_1$ and $z - bx_1$ respectively, we may assume that $a = b = 0$. If we set then

$$x'_1 = x_1 \quad , \quad x'_2 = \frac{x_2}{x_1} \quad , \quad z' = \frac{z}{x_1},$$

then $x'_1, x'_2, z'$ will be local coordinates at P′, and we will have $A_i(x'_1, x'_1 x'_2) = x_1'^i A'_i(x'_1, x'_2)$, where $A'_i(x'_1, x'_2)$ is a power series in $x'_1, x'_2$. We set

$$(2) \qquad f'(X'_1, X'_2, Z') = Z'^s + A'_1(X'_1, X'_2) Z'^{s-1} + \cdots + A'_s(X'_1, X'_2),$$

whence

$$f(X'_1, X'_1 X'_2, X'_1 Z') = X'^s f'(X'_1, X'_2, Z').$$

Then

$$(3) \qquad f'(x'_1, x'_2, z') = 0$$

is a local equation of F′, at P′ (and necessarily, $A'_s(0, 0) = 0$, since $f'(0, 0, 0) = 0$).

Suppose now that the multiplicity of P′ for F′ is still $s$ (by (2), it cannot be greater than $s$). Then two things must happen simultaneously: (1) $Z' = 0$ must be an $s$-fold root of the polynomial $f'(0, 0, Z')$, i.e., we must have $A'_i(0, 0) = 0$, for $i = 1, 2, \cdots, s$; (2) $x'_1, x'_2$ must be transversal local parameters at P′. The discriminant $D'(x'_1, x'_2)$ of F′, with respec to $z'$, is related to the discriminant $D(x_1, x_2)$ of $f$, with respect to $z$, by the following equation:

$$(4) \qquad D(x'_1, x'_1 x'_2) = x_1'^{s(s-1)} D'(x'_1, x'_2).$$

If then $\Delta'_{x'_1,x'_2}$ is the local critical curve of $F'$ at $P'$, associated with the parameters $x'_1, x'_2$, then it follows from (4) that $\Delta'_{x'_1,x'_2}$ is contained in the total transform of $\Delta_{x_1,x_2}$ under the quadratic transformation centered at the center $x_1 = x_2 = 0$ of $\Delta_{x_1,x_2}$.

Since the number of exceptional $s$-fold points of $F'$ which correspond to P is finite, and since it is known that after a finite number of locally quadratic transformations we get a total transform of $\Delta_{x_1,x_2}$ having only ordinary double points, the proposition is proved.

From now on we shall assume that all the exceptional singularities of F, of highest multiplicity $s$, are quasi-ordinary singularities of F.

## § 2. Analysis of quasi-ordinary multiple points.

Let P be a quasi-ordinary $s$-fold point of F, where $s = e(F)$, and let, then, $x_1$ and $x_2$ be local transversal parameters at P such that the local critical curve $\Delta_{x_1,x_2}$ has an ordinary double point at the origin $x_1 = x_2 = 0$. By a biholomorphic transformation of the local parameters $x_1, x_2$ we can arrange that $\Delta_{x_1,x_2}$ consists of the lines $x_1 = 0$ and $x_2 = 0$. It is then well known (3) (the ground field $k$ being algebraically closed and of characteristic zero) that each of the $s$ roots $z_\alpha$ of the defining equation (1) of F is a fractional power series in $x_1, x_2$:

$$(5) \qquad z_\alpha = \varphi_\alpha(x_1, x_2) \quad , \quad \varphi_\alpha(0, 0) = 0,$$

where by a fractional power series in $x_1, x_2$ we mean a power series in $x_1, x_2$ with rational, non-negative exponents, having bounded denominators.

By a fractional *monomial* we shall mean a monomial $x_1^{\alpha_1} x_2^{\alpha_2}$ where $\alpha_1$ and $\alpha_2$ are rational, non-negative numbers. Given two fractional monomials $M_1, M_2$, with $M_i = x_1^{\alpha_{i1}} x_2^{\alpha_{i2}}$ $(i = 1, 2)$, we say that $M_1$ divides $M_2$ if $\alpha_{1j} \leqq \alpha_{2j}$ for $j = 1, 2$, in other words: if the quotient $M_2 | M_1$ is a fractional monomial. Since the discriminant $D(x_1, x_2)$ of $f(x_1, x_2; Z)$ is, by assumption, of the form $x_1^{n_1} x_2^{n_2} \varepsilon(x_1, x_2)$, where $n_1$ and $n_2$ are positive integers and $\varepsilon(x_1, x_2)$ is a unit in the power series ring $k[[x_1, x_2]]$, it follows that we have

$$(6) \qquad z_\alpha - z_\beta = M_{\alpha\beta}\, \varepsilon_{\alpha\beta}(x_1, x_2), \qquad \alpha \neq \beta \quad ; \quad \alpha, \beta = 1, 2, \cdots, s,$$

where $M_{\alpha\beta}$ is a fractional monomial in $x_1, x_2$ and $\varepsilon_{\alpha\beta}(x_1, x_2)$ is a fractional power series such that $\varepsilon_{\alpha\beta}(0, 0) \neq 0$. Let M be the common divisor of the $s(s-1)/2$ monomials $M_{\alpha\beta}$:

$$M = x_1^{\lambda_1} x_2^{\lambda_2},$$

where $\lambda_1$ and $\lambda_2$ are therefore non-negative rational numbers.

(3) See, for instance, Abhyankar [1], Theorem 3.

*Proposition 2.1. There exists an integral power series $g(x_1, x_2)$ such that for each $\alpha = 1, 2, \cdots, s$ we have*

$$z_\alpha = g(x_1, x_2) + x_1^{\lambda_1} x_2^{\lambda_2} G_\alpha(x_1, x_2), \tag{7}$$

*where $G_\alpha(x_1, x_2)$ is a fractional power series and where for each index $\alpha$ there exists an index $\beta$ such that $G_\alpha(0, 0) \neq G_\beta(0, 0)$. Furthermore, we have*

$$\lambda_1 + \lambda_2 \geqq 1. \tag{8}$$

*Proof*: From the identity

$$M_{\alpha\beta}\,\varepsilon_{\alpha\beta} + M_{\beta\gamma}\,\varepsilon_{\beta\gamma} + M_{\gamma\alpha}\,\varepsilon_{\gamma\alpha} = 0 \tag{9}$$

which holds for any three indices $\alpha, \beta, \gamma$, follows that either $M_{\alpha\beta}$ divides $M_{\beta\gamma}$ or $M_{\beta\gamma}$ divides $M_{\alpha\beta}$. As a consequence, for any fixed $\alpha$ it is true that the $s-1$ monomial $M_{\alpha\beta}$ are completely ordered by the divisibility relation. If, then, we set, $M_\alpha$ = highest common divisor of the $M_{\alpha\beta}$ ($\beta = 1, 2, \cdots, s$; $\beta \neq \alpha$), then $M_\alpha$ is one of the monomials $M_{\alpha\beta}$, and we have, furthermore, that $x_1^{\lambda_1} x_2^{\lambda_2}$ is the highest common divisor of $M_1, M_2, \cdots, M_s$.

We assert that $M_1 = M_2 = \cdots = M_s$. For, let, say, $M_1 = M_{12}$, and let $\alpha, \beta$ be two distinct indices, different from 1; assume also that, say, $\alpha \neq 2$. We have that $M_{12}$ divides $M_{1\alpha}$. From the identity (9) follows that if, say, $M_{\alpha\beta}$ divides $M_{\beta\gamma}$, then $M_{\alpha\beta}$ also divides $M_{\gamma\alpha}$. *Hence* $M_1$ ($= M_{12}$) *divides* $M_{2\alpha}$, for all $\alpha \neq 2$ (including $\alpha = 1$, since $M_{12} = M_{21}$). Hence $M_1$ divides $M_2$. Similarly, $M_2$ must divide $M_1$. Hence $M_1 = M_2$, and similarly $M_1 = M_\alpha$ for $\alpha = 2, 3, \cdots, s$.

We go back to the power series $\varphi_\alpha(x_1, x_2)$ in (5) and we denote by $g(x_1, x_2)$ the sum of those terms of $\varphi_1(x_1, x_2)$ which are not divisible by $x_1^{\lambda_1} x_2^{\lambda_2}$. Thus, we can write

$$z_1 = g(x_1, x_2) + x_1^{\lambda_1} x_2^{\lambda_2} G_1(x_1, x_2),$$

where $G_1(x_1, x_2)$ is a fractional power series. We have, for each $\alpha \neq 1$: $z_\alpha = z_1 + (z_\alpha - z_1) = g(x_1, x_2) + x_1^{\lambda_1} x_2^{\lambda_2} G_1(x_1, x_2) + M_{\alpha 1}\,\varepsilon_{\alpha,1}(x_1, x_2)$. Since $x_1^{\lambda_1} x_2^{\lambda_2}$ divides $M_{\alpha 1}$, this last expression of $z_\alpha$ is indeed of the form (7). For fixed $\alpha$ we have $z_\alpha - z_\beta = x_1^{\lambda_1} x_2^{\lambda_2}(G_\alpha(x_1, x_2) - G_\beta(x_1, x_2))$. Since we know that for some $\beta$ we must have $M_{\alpha\beta} = M_\alpha = x_1^{\lambda_1} x_2^{\lambda_2}$, it follows from (6) that for that particular $\beta$ (whose value depends on $\alpha$) we must have $G_\alpha(x_1, x_2) - G_\beta(x_1, x_2) = \varepsilon_{\alpha\beta}(x_1, x_2)$, whence $G_\alpha(0, 0) \neq G_\beta(0, 0)$.

To prove that $g(x_1, x_2)$ is an *integral* power series, assume the contrary. Let

$$\varphi_1(x_1, x_2) = \sum_{i,j} c_{ij}\, x_1^{i/m}\, x_2^{j/m}, \qquad (c_{ij} \in k)$$

and let $x_1^{p/m} x_2^{q/m}$ be a monomial, *not divisible* by $x_1^{\lambda_1} x_2^{\lambda_2}$, such that $c_{pq} \neq 0$, and assume that at least one of the exponents $p/m$, $q/m$ is not an integer. It $\omega$ denotes a primitive $m^{\text{th}}$ root of unity and $a, b$ are arbitrary integers, then the power series

$$\psi(x_1, x_2) = \sum_{i,j} c_{ij}\, \omega^{ai+bj}\, x_1^{i/m}\, x_2^{j/m}$$

represents one of the conjugates of $z_1$ and is therefore equal to one of the power series $\varphi_\alpha(x_1, x_2)$ $(\alpha = 1, 2, \cdots, s)$. Let $d$ be the highest common divisor of $p$ and $q$. We choose the integers $a$ and $b$ so as to have $ap + bq = d$. Since $d \not\equiv 0 \pmod m$ it follows that the coefficient $c_{pq}\,\omega^d$ of the monomial $x_1^{p/m} x_2^{q/m}$ is different from $c_{pq}$. Hence $\psi(x_1, x_2) = \varphi_\alpha(x_1, x_2)$, with $\alpha \neq 1$, and the monomial $x_1^{p/m} x_2^{q/m}$ actually occurs in the power series $\varphi_\alpha(x_1, x_2) - \varphi_1(x_1, x_2)$. Therefore $M_{1\alpha}$ must divide $x_1^{p/m} x_2^{q/m}$, and hence–*a fortiori*– $x_1^{\lambda_1} x_2^{\lambda_2}$ must divide $x_1^{p/m} x_2^{q/m}$ (since $x_1^{\lambda_1} x_2^{\lambda_2} = M_1 = M_2 = \cdots = M_s$). This is in contradiction with our choice of the monomial $x_1^{p/m} x_2^{q/m}$.

Finally, to prove (8), we observe that

$$f(x_1, x_2, Z) = \prod_{\alpha=1}^{s} (Z - \varphi_\alpha(x_1, x_2)) = \tag{10}$$

$$= \prod_{\alpha=1}^{s} (Z - g(x_1, x_2) - x_1^{\lambda_1} x_2^{\lambda_2} G_\alpha(x_1, x_2)).$$

Since the origin P is an $s$–field point of F, the terms of least degree in $f$ must be of degree $s$. Now each of the $s$ factors $Z - \varphi_\alpha(x_1, x_2)$ contains terms of degree 1 (for istance, the term Z). Hence, in each of these factors the terms of lowest degree *must be of degree* 1. Now, every term in $g(x_1, x_2)$ is of degree $\geqq 1$ (since the relation $f(0, 0, 0) = 0$ implies $g(0, 0) = 0$). Hence, for each $\alpha = 1, 2, \cdots, s$, it is true that in the fractional power series $x_1^{\lambda_1} x_2^{\lambda_2} G_\alpha(x_1, x_2)$ the terms of lowest degree must be of degree $\geqq 1$. From the fact that $G_\alpha(0, 0) \neq G_\beta(0, 0)$ for some pair of indices $\alpha, \beta$, follows that $G_\alpha(0, 0) \neq 0$ for some index $\alpha$. This implies that $\lambda_1 + \lambda_2 \geqq 1$, and completes the proof.

*Proposition* 2.2. *Let* P *be a quasi-ordinary s–field point of* F *and let* (1) *be the local equation of* F *at* P, *where we assume that* $x_1, x_2$ *are local transversal parameters and that the critical curve* $\Delta_{x_1,x_2}$ *consists of the two lines* $x_1 = 0$ *and* $x_2 = 0$. *Let* $\Gamma_i$ $(i = 1, 2)$ *denote the (locally) irreducible algebroid curve through* P, *defined by the equations*

$$\Gamma_i : x_i = 0 \quad , \quad z + \frac{A_1(x_1, x_2)}{s} = 0. \tag{11}$$

*Then* $\Gamma_1$ *and* $\Gamma_2$ *are the only possible locally irreducible s–fold curves of* F *through* P, *and* $\Gamma_i$ *is s–fold for* F *if and only if* $\lambda_i \geqq 1$. *Furthermore, if* W *is any irreducible s–fold curve of* F *passing through* P *(whence the local component of* W *at* P *is either one of the curves* $\Gamma_1, \Gamma_2$ *or their union) and if the monoidal transformation* $T : F' \to F$ *with center* W *has the property that the (necessarily finite) set* $T^{-1}\{P\}$ *contains a point* P′ *which is still s–fold for* F′, *then* $T^{-1}\{P\} = P'$ *and* P′, *if exceptional for* F′, *is a quasi–ordinary singularity of* F′.

*Proof*: Let $\pi$ be the projection of F onto the $(x_1, x_2)$–plane defined generically by $\pi(x_1, x_2, z) = (x_1, x_2)$. Let $\Gamma$ be an irreducible algebroid curve on F through P which is singular for F. Then $\pi(\Gamma)$ must be contained in $\Delta_{x_1,x_2}$; in other words, $\Gamma$ is contained in one of the planes $x_1 = 0$, $x_2 = 0$. Let, say, $x_1 = 0$ on $\Gamma$. The generic point of $\Gamma$ has then coordinates $X_1 = 0$,

$X_2 = \xi$, $Z = \zeta$, where $\xi$ is a trascendental over $k$ and where $\zeta$ is a root of the polynomial $f(0, \xi, Z)$ in Z. If $\Gamma$ is $s$–fold, then $\zeta$ must be an $s$–fold of this polynomial (which has degree $s$ in Z). Hence, in this case, we must have $f(0, \xi, Z) = (Z - \zeta)^s$, showing that $\zeta = -\frac{A_1(0, \xi)}{s}$. In other words, $\Gamma$ is necessarily the curve $\Gamma_1$ defined in (11).

Upon replacing $z$ by $z + \frac{A_1(x_1, x_2)}{s}$ we may assume that $A_1(x_1, x_2)$ is zero and that consequently $\Gamma_i$ is the line $x_i = z = 0$. This change of $z$ affects only the power series $g(x_1, x_2)$ in (7), but not the integers $\lambda_1, \lambda_2$.

Assume that $\Gamma_1$, say, is indeed $s$–fold for F. Each term of the power series $f(x_1, x_2, Z)$ must then be of degree $\geqq s$ in $x_1, Z$. Using the factorization (10) of $f$ we deduce that each of the $s$ fractional power series $g(x_1, x_2) + x_1^{\lambda_1} x_2^{\lambda_2} G_\alpha(x_1, x_2)$ must be free from terms of degree $< 1$ in $x_1$. Since $g(x_1, x_2)$ is an integral power series and $\lambda_1 + \lambda_2 \geqq 1$, it follows that $\lambda_1 \geqq 1$. Conversely, assume that $\lambda_1 \geqq 1$. Then each factor $Z - \varphi_\alpha(x_1, x_2)$ in (10) is of degree $\geqq 1$ in $x_1, Z - g(x_1, x_2)$. Hence the curve $\Gamma$ defined by $x_1 = Z - g(x_1, x_2) = 0$ is $s$–fold for F. This implies that $g(0, \xi) = -\frac{A_1(0, \xi)}{s}$, and thus $g(0, \xi) = 0$ (since we have assumed that $A_1(x_1, x_2)$ is zero). Hence $g(x_1, x_2)$ is divisible by $x_1$, and consequently the curve $\Gamma$ coincides with the curve $\Gamma_1 : x_1 = z = 0$, which is thus $s$–fold for F.

Let now W and $T : F' \to F$ be as in the proposition. If W consists, locally at P, of both curves $\Gamma_1$ and $\Gamma_2$, T is a product of two monoidal transformations with locally irreducible $s$–fold curves as center. Hence, to complete the proof of the proposition, it is sufficient to consider the case in which W is locally irreducible at P, say $W = \Gamma_1 : x_1 = z = 0$. We know that in this case $g(x_1, x_2)$ is divisible by $x_1$ and that $\lambda_1 \geqq 1$. Hence $z/x_1$ is an integral function of $x_1, x_2$, showing that if P′ is any point of $T^{-1}\{P\}$ then $z/x_1$ belongs to the local ring of P′ on F′.

A set of local coordinates at P′ (i.e., a basis of the maximal ideal of the local ring of P′) is then given by $x_1, x_2, z'$, where $z' = \frac{z}{x_1} - c$ and $c$ is the constant term $c_\alpha$ of one of the fractional power series

$$g(x_1, x_2)/x_1 + x_1^{\lambda_1 - 1} x_2 G_\alpha(x_1, x_2).$$

Now, if we set $f(x_1, x_2, x_1(z' + c)) = x_1^s f'(x_1, x_2, z')$, and if we assume that P′ is an $s$–fold point of F′, then 0 must be an $s$–fold root of the polynomial $f'(0, 0, Z')$ (which has degree $s$ in Z′). It follows that in this case all the $c_\alpha$ are equal to $c$, showing that $T^{-1}\{P\}$ consists of the single point $P' = (0, 0, 0)$. Since P′ is $s$–fold for F′ and since 0 is an $s$–fold root of $f'(0, 0, Z')$, it follows that $x_1, x_2$ are local transversal parameters of P′ on F′ ($f(x_1, x_2, Z') = 0$ being a local equation of F′ at P′). If $D(x_1, x_2)$ and $D'(x_1, x_2)$ are respectively the discriminants of $f(x_1, x_2, Z)$ (with respect to Z) and of $f'(x_1, x_2, Z')$ (with respect to Z′), then we have

$$D = x_1^{\frac{s(s-1)}{2}} D'.$$

This shows that the critical curve $\Delta'_{x_1,x_2}$ of F′ (at P′) is contained in $\Delta_{x_1,x_2}$, i.e., is contained in the union of the two lines $x_1 = 0$, $x_2 = 0$. Hence, if P′ is an exceptional singularity, $\Delta'_{x_1,x_2}$ is the set of both lines $x_1 = 0$, $x_2 = 0$, and thus P′ is a quasi-ordinary multiple point of F′. This completes the proof of the proposition.

COROLLARY 2.3. *If all the exceptional singularities of* F, *of highest multiplicity* $s$ $(= e(\mathrm{F}))$ *are quasi-ordinary, then any monoidal transformation of* F *whose center is an s-fold curve* W *of* F *is permissible.*

For, in the first place, no point of W can have multiplicity greater than $s$ for F, for in the contrary case any such point would be an exceptional singularity of F, contrary to the assumption that $s = e(\mathrm{F})$. In the second place, if P is a singular point of W, then P is necessarily an exceptional singularity of F, and since P has highest multiplicity $s$ it is quasi ordinary, and thus, by the preceding proposition, P can only be an ordinary double point of W.

*Definition* 2.4. *A singular point* P *of* F, *of multiplicity s, is said to be strictly exceptional if* P *does not lie on any s-fold curve of* F.

COROLLARY 2.5. *A quasi-ordinary multiple point* P *of* F *is strictly exceptional if and only if* $\lambda_1 < 1$ *and* $\lambda_2 < 1$.

This is a direct consequence of Proposition 2.2.

*Proposition* 2.6. *Let* P *be a strictly exceptional, quasi-ordinary multiple point of* F, *of multiplicity* $s$ $(= e(\mathrm{F}))$, *let* $\lambda(\mathrm{P}) = \lambda_1 + \lambda_2$ *and let* $\mathrm{T} : \mathrm{F}' \to \mathrm{F}$ *be the locally quadratic transformation of* F *with center* P. *If* $\lambda(\mathrm{P}) > 1$ *then* $\mathrm{T}^{-1}\{\mathrm{P}\}$ *contains at most two points which have multiplicity s for* F′; *if* P′ *is one of these points then* P′ *is quasi-ordinary, strictly exceptional and* $\lambda(\mathrm{P}') < \lambda(\mathrm{P})$.

*Proof*: Let P′ be any point of $\mathrm{T}^{-1}\{\mathrm{P}\}$. Since $x_1, x_2$ are local transversal parameters at P, we may assume (see proof of Proposition 1.2) that $x_2/x_1$ and $z/x_1$ belong to the local ring of P′ on F′ and that, consequently, there exists constants $a, b$ in $k$ such that $x_1, x'_2 = \dfrac{x_2}{x_1} - a$ and $z' = \dfrac{z}{x_1} - b$ generate the maximal ideal of the local ring of P′. We may replace $z$ by $z + g(x_1, x_2)$ and we may therefore assume that $g(x_1, x_2)$ is zero. Then we will have

$$f'(x_1, x'_2, \mathrm{Z}') = \prod_{\alpha=1}^{s} [\mathrm{Z}' + b - x_1^{\lambda_1+\lambda_2-1}(x'_2 + a)^{\lambda_2} \mathrm{G}_\alpha(x_1, x_1(x'_2 + a))],$$

where $f'(x_1, x'_2, \mathrm{Z}') = x_1^s f(x_1, x_1(x'_2 + a), x_1(\mathrm{Z}' + b))$, and $f'(x_1, x'_2, \mathrm{Z}') = 0$ is the local equation of F′ at P′.

Each of the $s$ factors on the right hand side contains terms of degree 1 (for instance Z′), and if $\lambda_1 + \lambda_2 = 1$ and $\alpha$ is an index such that $\mathrm{G}_\alpha(0, 0) \neq 0$, then it is true that the associated factor contains a term of degree *less* than 1, namely the term $x^{\lambda_2} \mathrm{G}_\alpha(0, 0)$ (since, by Corollary 2.5, we have $\lambda_2 < 1$). Hence in this case, $f'$ contains terms of degree less than $s$, and P′ is of multiplicity less than $s$.

Assume now that $\lambda_1 + \lambda_2 > 1$. For P′ to be an $s$–fold point of F′ it is necessary that $Z' = 0$ be an $s$–fold root of $f'(0, 0, Z')$. Now, $f'(0, 0, Z') = (Z' + b)^s$ (since $\lambda_1 + \lambda_2 > 1$). Hence we must have $b = 0$. *We assert that also* $a = 0$. For, were $a \neq 0$, then for the index $\alpha$ such that $G_\alpha(0, 0) \neq 0$ the associated factor $Z' - x_1^{\lambda_1+\lambda_2-1}(x_2' + a)^{\lambda_2} G_\alpha(x_1, x_1(x_2' + a))$ would contain the term $x_1^{\lambda_1+\lambda_2-1} a^{\lambda_2} G_\alpha(0, 0)$ of degree less than 1 (since $\lambda_1 < 1$ and $\lambda_2 < 1$), and P′ could not be $s$–fold. Thus both $a$ and $b$ are zero. What we have shown is that there are only two points of $T^{-1}\{P\}$ which could possibly be $s$–fold points of F′, namely the points in which either $x_2/x_1$ and $z/x_1$ or $x_1/x_2$ *and* $z/x_2$ have zero residues.

If P′ is such a point which is $s$–fold for F′, and if say, $x_2' = \frac{x_2}{x_1}$ and $z' = \frac{z}{x_1}$ have zero residues at P′, then we find (as in the proof of Proposition 1.2) that the local critical curve $\Delta'_{x_1, x_2'}$ of $F'$ at $P'$ is contained in the union of the two lines $x_1 = 0$, $x_2' = 0$. Since P′ is $s$–fold *but does not lie on any s–fold curve* (as P does not lie on any $s$–fold curve and as no irreducible component of $T^{-1}\{P\}$ is $s$–fold for F′), it follows that P′ is a strictly exceptional, quasi-ordinary multiple point of F′. Furthermore, if we set $z'_\alpha = \frac{z_\alpha}{x_1}$ then we find that

$$z'_\alpha - z'_\beta = x_1^{\lambda_1+\lambda_2-1} x_2'^{\lambda_2} [G_\alpha(x_1, x_1 x_2') - G_\beta(x_1, x_1 x_2')] = M'_{\alpha\beta} \varepsilon'_{\alpha\beta}(x_1, x_2'),$$

with $\varepsilon'_{\alpha\beta}(0, 0) \neq 0$. Since for some $\alpha, \beta$ we have $G_\alpha(0) - G_\beta(0) \neq 0$, it follows that the highest common divisor of the monomial $M'_{\alpha\beta}$ is $x_1^{\lambda_1+\lambda_2-1} x_2'^{\lambda_2}$. Thus $\lambda_1' = \lambda_1 + \lambda_2 - 1$, $\lambda_2' = \lambda_2$ and $\lambda(P') = \lambda_1' + \lambda_2' = \lambda_1 + \lambda_2 + (\lambda_2 - 1) = \lambda(P) + (\lambda_2 - 1) < \lambda(P)$, since $\lambda_2 < 1$.

## § 3. ELIMINATION OF THE QUASI-ORDINARY $s$–FOLD POINTS ($s = e(F)$).

The proof can now be rapidly concluded. We start with a surface on which all the exceptional singularities of highest multiplicity $s$ ($= e(F)$) are already quasi-ordinary. Our first step is to apply to F a monoidal transformation $T : F' \to F$ centered at an irreducible $s$–fold curve $\Gamma$, *provided* $\Gamma$ carries exceptional $s$–fold points. By Corollary 2.3, such a transformation is permissible. *If* $e(F')$ is still equal to $s$, all the exceptional $s$–fold points of F′ are still quasi-ordinary (Proposition 2.2). If F′ still carries an irreducible $s$–fold curve $\Gamma'$ which contains exceptional $s$–fold points, we again apply a monoidal transformation $F'' \to F'$ centered at $\Gamma'$. Since the monoidal transformations used in this step do not blow up any points (F′, F″, etc. are dominated by the normalization of F), the case in which $e(F) = e(F') = \cdots = e(F^{(i)}) = \cdots$ *and* each $F^{(i)}$ contains an exceptional $s$–fold point which is not strictly exceptional cannot arise indefinitely. Hence, after a finite number of steps we will ultimately get a surface $F_1$ for which *either* $e(F_1) < s$ *or* $e(F_1) = s$ and all exceptional $s$–fold points of $F_1$ are strictly exceptional (this latter case may present itself even *before* the $s$–fold curves of F have been eliminated).

Assume now that already on F we have the situation in which all the exceptional $s$–fold points are quasi-ordinary and strictly exceptional. At this stage we begin to apply locally quadratic transformations centered at exceptional $s$–fold points. Proposition 2.6 shows that after a finite number of such transformations we will get a surface $F_2$ such that $e(F_2) < s$ (since $\lambda(P) \geqq 1$ at every quasi-ordinary $s$–fold point).

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